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Variational Noether's Theorem: the Interplay of Time, Space and Gauge

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Abstract

This paper revisits Noether's theorem on the constants of motion for Lagrangian mechanical systems in the ODE case, attempting a clarification on both the theoretical and the applied side. Noether's variational theorem requires some form of (infinitesimal) invariance of the Lagrangian with respect to some set of transformations, and provides conserved quantities as a result. First of all, we obtain both a simpler theory and new applications by allowing transforms that are not point functions. Then we compare the three known formulations of Noether's theorem, that involve respectively (1) invariance without gauge transform, under both dependent and independent variable transformation; (2) gauge-invariance under a transformation of dependent variable; (3) gauge-invariance under transformation of both dependent and independent variable. We show that, in the case of one independent variable, all three formulations are equivalent, in the sense that any conservation law, that can be deduced with one, can also be deduced with any other. In the application sections we work out several examples following a unified general scheme and using some newly devised transformations, most notably in the derivation of the Laplace-Runge-Lenz vector for Kepler's problem.

1 Introduction

Emmy Noether's classical 1918 work [8] proved that conservation laws in variational Mechanics follow whenever the Lagrangian function is *invariant* under a one-parameter continuous group that *transforms* both *dependent* and *independent variables*. This result unifies under a single idea many old conservation laws, suggests a way of discovering new ones, and imposes theoretical constraints on how to build new Lagrangian theories. Since then, the literature has seen a vast number of formulations, of various degrees of generality, all going under the rubric of Noether's theorem.

One generalization is so easy that we are still going to refer to it as the "original" Noether's setting: it consist in relaxing exact finite invariance to *infinitesimal* invariance, and relaxing the group of transformation to a mere *family* of transformations. Noether kept her famous paper in the group framework, probably because her other major focus was on a form of converse theorem.

As Bluman and Kumei recount in the historical notes of their book [3], Bessel-Hagen already in 1921 [2] observed that infinitesimal invariance of the Lagrangian can be replaced by the (apparently) more general invariance up to a divergence, or, as we prefer to call it, *gauge-invariance*, under both dependent and independent variable transformation.

Then Boyer [5] in 1967 noticed that Bessel-Hagen's formulation was unnecessarily complicated, because one could get the same first integrals by means of gauge-invariance under transformation of the dependent variable only.

Textbooks that feel like dwelling on Noether's theorem as little as possible usually do not touch the independent variable, and assume either gauge-less invariance (Arnold's book [1], for example) or, at most, gauge-invariance, as recommended by Lévy-Leblond in 1970 [7]. Giaquinta and Hildebrandt [6] develop the theory in the PDE case, first with independent and dependent variable change, and then they add a version with invariance up do a divergence.

In this paper we only deal with the case of one independent variable, which we call *time*, while the dependent variable(s) will be called *space*. In this simpler setting we have tried to rethink and systematize both the theory and the applications of Noether's theorem.

Section 2 is devoted to general underpinnings and motivations. We will make precise what we mean with the transformations and invariances. We have tried to isolate the most general setting where we are able to deduce constants of motions. We define a generalized notion of gauge invariance, in terms of what we will call "gauge term" or simply "gauge". We do not assume from the outset that space change, time change and gauge are point functions of time, space, velocity and the parameter: the advantage is that the theory is cleaner this way, and we leave open the chance that those objects may be not point functions, but, say, integral or delay functionals. In the methodological notes we describe the abstract scheme that we will follow in the examples later on.

In Section 3 we turn to the issue of which invariance is more general than the others. Our contribution is that the original Noether's version is actually no less general than Bessel-Hagen's and Boyer's versions. Schematically, we prove the equivalence of the following three formulations:

1. invariance under time and space changes (original Noether).
2. gauge-invariance under space changes (Boyer);

3. gauge-invariance under both time and space changes (Bessel-Hagen);

where invariance is in the infinitesimal sense. By equivalence we mean that whenever one formulations holds for a system, then any other holds too, although with a possibly different gauge or time change, and they all lead to the same first integral. The precise statement is in Section 3 below (Theorems 1 and 2).

In Section 4 we apply the general method to the most basic textbook applications of Noether's theorem: invariance by time-shift, space translation and space rotation. In particular we give a fresh look at conservation of energy, with an integral functional as a gauge. In Section 5 we try out various approaches to the very simple system of the free fall of a weight.

In Section 6 we deduce Laplace-Runge-Lenz's invariant for Kepler's system using an innovative, simpler space-change family, that uses a delay term $q(t+\varepsilon)$, in contrast to the traditional point function of $q(t), \dot{q}(t)$ that is affine in ε. Then, using some variation on the theme, we show that, in principle, Noether's theorem may apply to single motions, and not necessarily to all motions at once.

Section 7 gives a Noetherian deduction of the first integrals of a family of superintegrable systems that have recently been discovered by different methods [9]. Finally, in Section 8 we give a reworking of an example that we found in a 1988 paper [4] by Bobillo-Ares. In both Section 7 and 8 we again use gauge terms that do not fit directly into the customary "gauge transform" concept.

Notations

Throughout this work, $\mathbb{R}^n$ will be the usual Euclidean space, and we will use the notation $x \cdot y$ for the inner product and $\|x\|$ for the norm. The symbol q will denote either a vector in $\mathbb{R}^n$ or an $\mathbb{R}^n$-valued function of une variable, as the context should clarify which is the case. Given two points $x = (x_1, x_2)$, $y = (y_1, y_2)$ in the plane $\mathbb{R}^2$ we set $\det(x, y) := x_1 y_2 - x_2 y_1$. To make the statements less cumbersome we will assume that our smooth functions $L, G, \tau \ldots$ are defined not on some open sets but on whole Euclidean spaces, and leave the obvious generalizations to the reader. For partial derivatives of expressions with respect to a variable x we use the fractional notation $\frac{\partial}{\partial x}$, whilst for the partial derivatives of a named function F we write more simply $\partial_x F(x, y)$ or $\partial^2_{x,y} F(x, y)$. The gradient of a smooth scalar function $q \mapsto f(q)$ of n variables will be denoted by $\partial_q f(q)$, and it will be treated as a vector in $\mathbb{R}^n$. To clarify our usage without a formal definition, here is Taylor's formula for a scalar function of $(p, q) \in \mathbb{R}^n \times \mathbb{R}^n$:

$$f(p + h, q + k) = f(p, q) + \partial_p f(p, q) \cdot h + \partial_q f(p, q) \cdot k + o\big(\|p\| + \|q\|\big) .$$

2 Variational invariances

The variational approach to classical mechanics starts with a smooth Lagrangian function $L\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^N\to\mathbb{R}$, upon which the action functional is defined as

$$A_{a,b}(q) := \int_a^b L\big(t, q(t), \dot q(t)\big)\, dt\,. \tag{1}$$

We then posit that the mechanical motions will be fixed-endpoint-stationary for the action functional. Hamilton's variational principle translates the stationarity condition into the Lagrange differential equations:

$$\frac{d}{dt}\partial_{\dot q} L\big(t, q(t), \dot q(t)\big) - \partial_q L\big(t, q(t), \dot q(t)\big) = 0 \qquad \forall t\,. \tag{2}$$

Noether's contribution was to show that mechanical conservation laws can be deduced from some kind of invariance property of the action functional. Let us review these invariances, starting from the simplest case.

Suppose we have a smooth trajectory $q(t)$ that for now may not necessarily be a solution to the Lagrange equations. Let us embed the trajectory into a one-parameter family of trajectories $q_\varepsilon(t)$ (*space change*), so that $q_0(t)\equiv q(t)$. We can visualize the family with the help of Figure 1. Notice that there is no need for the endpoints $q_\varepsilon(a), q_\varepsilon(b)$ to be fixed. In the picture the joint map $(\varepsilon, t)\mapsto q_\varepsilon(t)$ is one-to-one and with maximum rank, but this is only arranged to make things readable, and it is not at all required for the theory.

Consider the action functional along the family as a function of ε:

$$\varepsilon \mapsto A_{a,b}(q_\varepsilon)\,. \tag{3}$$

In Figure 1 it corresponds to the integral of L along the thicker arcs. One crucial and easy fact is that the derivative of the action with respect to ε at $\varepsilon = 0$ has an integral-free formula if we assume that $q(t)$ is a solution to the Lagrange equations:

$$\begin{aligned}\frac{\partial A_{a,b}(q_\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0} &= \partial_{\dot q} L\big(\xi, q(\xi), \dot q(\xi)\big)\cdot\partial_\varepsilon q_\varepsilon(\xi)\Big|_{\varepsilon=0,\xi=b} - \\ &\quad - \partial_{\dot q} L\big(\xi, q(\xi), \dot q(\xi)\big)\cdot\partial_\varepsilon q_\varepsilon(\xi)\Big|_{\varepsilon=0,\xi=a}\end{aligned} \tag{4}$$

which is of the form

$$\frac{\partial A_{a,b}(q_\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0} = F(b) - F(a) \tag{5}$$

where we have set

$$F(t) = \partial_{\dot q} L\big(t, q(t), \dot q(t)\big)\cdot\partial_\varepsilon q_\varepsilon(t)\big|_{\varepsilon=0}\,. \tag{6}$$

The basic idea is that if it somehow also happens that

$$\frac{\partial A_{a,b}(q_\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0} = 0 \qquad \forall a, b\,, \tag{7}$$

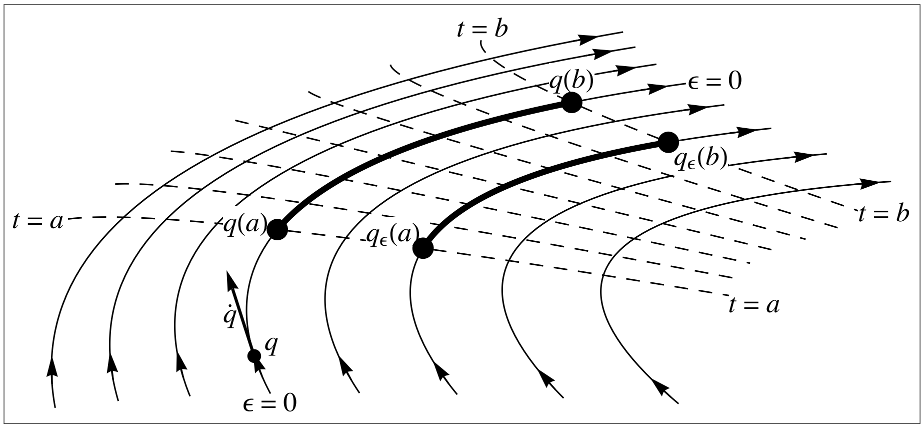

Figure 1: The continuous lines are the space change family $t \mapsto q_\varepsilon(t)$. The dashed lines are $\varepsilon \mapsto q_\varepsilon(t)$. The integral of formula (3) is performed in dt over the thicker arcs.

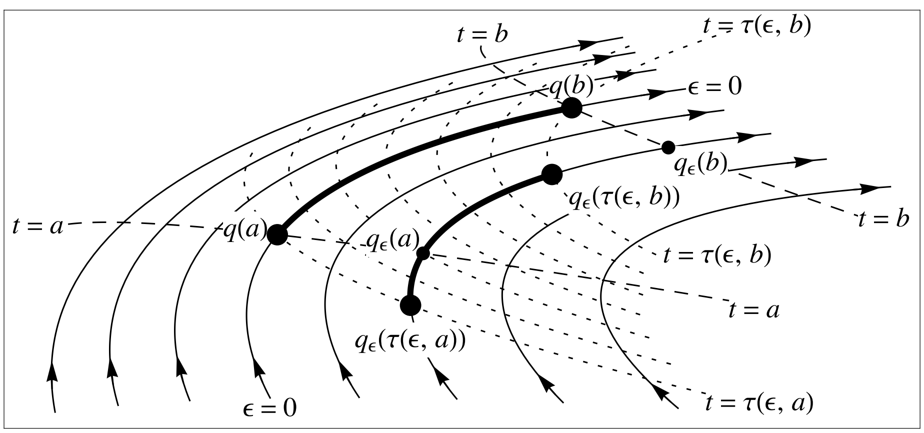

Figure 2: The continuous lines are the space change family $t \mapsto q_\varepsilon(t)$. The dotted lines are $\varepsilon \mapsto q_\varepsilon(\tau(\varepsilon, t))$. The integral of formula (13) is performed still in dt over the thicker arcs.

then the function $F(t)$ will be constant in t, which is the kind of result we want. When equation (7) is true, we will speak of *infinitesimal invariance under space change*. Notice that equation (7) can be rewritten in differential form as

$$\frac{\partial}{\partial\varepsilon} L\big(t, q_\varepsilon(t), \dot{q}_\varepsilon(t)\big)\Big|_{\varepsilon=0} = 0 \qquad \forall t\,. \tag{8}$$

There are situations when not only the function $\varepsilon \mapsto A_{a,b}(q_\varepsilon)$ has zero derivative at $\varepsilon = 0$, but it is actually constant, or, equivalently, $L(t, q_\varepsilon(t), \dot{q}_\varepsilon(t))$ does not depend on ε. These will be called *finite invariances under space change*. The most classic examples are the Lagrangians that are invariant under either translations or rotations (Examples 2 and 3 below), whereby any smooth trajectory $q(t)$ can be embedded in a translated or rotated family $q_\varepsilon(t)$ with the property of finite invariance. When $q(t)$ also solves the Lagrange equations, the momentum or angular momentum will be conserved respectively. Although very simple and neat, this concept of finite invariance does not seem powerful enough to cover conservation of energy, for instance.

We will generalize in two independent steps. First, we can modify the function (3) by introducing a smooth real function $G(\varepsilon, t)$, that for the purposes of this paper we have chosen to call *gauge*, and then ask that the new function

$$\varepsilon \mapsto A_{a,b}(q_\varepsilon) + G(\varepsilon, b) - G(\varepsilon, a) \tag{9}$$

has zero derivative at $\varepsilon = 0$, for all a, b. In terms of the Lagrangian, this is expressed as

$$\frac{\partial}{\partial\varepsilon}\Big(L\big(t, q_\varepsilon(t), \dot{q}_\varepsilon(t)\big) + \partial_t G(\varepsilon, t)\Big)\Big|_{\varepsilon=0} = 0\,. \tag{10}$$

This generalized condition will be called infinitesimal *gauge-invariance under space change*. Combined with (5), this invariance causes the modified function

$$t \mapsto F(t) + \partial_\varepsilon G(0, t) \tag{11}$$

to be a constant of motion, when $q(t)$ solves the Lagrange equations. Of course we may speak of *finite* gauge-invariance in the rare instances when the function (9) is constant (Example 1)

The simplest meaningful example of infinitesimal gauge-invariance occurs when the Lagrangian function L does not depend on t. We can embed any smooth $q(t)$ into its time-shift family $q_\varepsilon(t) := q(t + \varepsilon)$, whose derivatives with respect to t are the same as the derivatives with respect to ε. Let us try a gauge G of the form $G(\varepsilon, t) = \varepsilon g(t)$. The gauge-invariance condition (10) becomes

$$g'(t) = -\Big(\frac{\partial}{\partial\varepsilon} L\big(q_\varepsilon(t), \dot{q}_\varepsilon(t)\big)\Big)\Big|_{\varepsilon=0} = -\frac{d}{dt} L\big(q(t), \dot{q}(t)\big)\,. \tag{12}$$

By inspection we see that (10) holds with the choice $G(\varepsilon, t) := -\varepsilon \cdot L(q(t), \dot{q}(t))$. We can deduce that when $q(t)$ solves the Lagrange equations, we have *conservation of energy* (Example 1).

To add one more layer of complication, we modify the function (9), in its turn, by letting the endpoints of the integration to depend on ε through a smooth function $\tau(\varepsilon, t)$:

$$\begin{aligned} f_{a,b}(\varepsilon) &:= A_{\tau(\varepsilon,a),\tau(\varepsilon,b)}(q_\varepsilon) + G(\varepsilon, b) - G(\varepsilon, a) = \\ &= \int_{\tau(\varepsilon,a)}^{\tau(\varepsilon,b)} L\big(\xi, q_\varepsilon(\xi), \dot{q}_\varepsilon(\xi)\big)\, d\xi + G(\varepsilon, b) - G(\varepsilon, a)\,, \end{aligned} \tag{13}$$

with the compatibility condition that $\tau(0, t) \equiv t$. The function $\tau(\varepsilon, t)$ will be called *time change* in this paper. An attempt at visualization is in Figure 2. If $f'_{a,b}(0) = 0$ we will talk of *infinitesimal gauge-invariance under space and time change*. This condition also leads to a conservation law (Theorem 2) when, as usual, $q(t)$ is a solution to Lagrange equations. There is an example (Section 8) where q_ε, G, τ are all nontrivial. More commonly, though, the gauge will be missing (i.e., $G \equiv 0$) and $\tau(\varepsilon, t) \not\equiv t$, a situation that we will call *infinitesimal invariance with space and time change*.

To reformulate the condition $f'_{a,b}(0) = 0$ in terms of L, let us perform the change of variable $\xi = \tau(\varepsilon, t)$ for $t \in [a, b]$ in equation (13), that brings us to a fixed interval:

$$\begin{aligned} f_{a,b}(\varepsilon) := \int_a^b & L\big(\tau(\varepsilon, t), q_\varepsilon(\tau(\varepsilon, t)), \dot{q}_\varepsilon(\tau(\varepsilon, t))\big)\partial_t \tau(\varepsilon, t)\, dt + \\ &+ G(\varepsilon, b) - G(\varepsilon, a)\,. \end{aligned} \tag{14}$$

We can take the derivative with respect to ε under the integral sign, obtaining

$$f'_{a,b}(\varepsilon) = \int_a^b \frac{\partial}{\partial \varepsilon}\Big(L\big(\tau(\varepsilon, t), q_\varepsilon(\tau(\varepsilon, t)), \dot{q}_\varepsilon(\tau(\varepsilon, t))\big)\partial_t \tau(\varepsilon, t)\Big)\, dt + \tag{15}$$

$$+ \frac{\partial}{\partial \varepsilon}\big(G(\varepsilon, b) - G(\varepsilon, a)\big) = \tag{16}$$

$$= \int_a^b \frac{\partial}{\partial \varepsilon}\Big(L\big(\tau(\varepsilon, t), q_\varepsilon(\tau(\varepsilon, t)), \dot{q}_\varepsilon(\tau(\varepsilon, t))\big)\partial_t \tau(\varepsilon, t)\Big)\, dt + \tag{17}$$

$$+ \int_a^b \frac{\partial^2}{\partial \varepsilon \partial t} G(\varepsilon, t) dt\,. \tag{18}$$

Asking that $f'_{a,b}(0)$ vanish for all choices of a, b is equivalent to the formula

$$\frac{\partial}{\partial \varepsilon}\Big(L\big(\tau(\varepsilon, t), q_\varepsilon(\tau(\varepsilon, t)), \dot{q}_\varepsilon(\tau(\varepsilon, t))\big)\partial_t \tau(\varepsilon, t) + \partial_t G(\varepsilon, t)\Big)\Big|_{\varepsilon=0} \equiv 0\,. \tag{19}$$

For example, consider again a Lagrangian function L that does not depend on t, and take the same space change $q_\varepsilon(t) := q(t + \varepsilon)$ as before, but choose the null gauge $G(\varepsilon, t) \equiv 0$ and the new time change $\tau(t, \varepsilon) := t - \varepsilon$ instead. It is trivial to verify that $f_{a,b}(\varepsilon)$ does not depend on ε. This is a neat instance of *finite invariance with space and time change*. The induced first integral is the energy, again. Conservation of energy is the classical prototype of a conservation law that can be obtained in two different ways: (1) with gauge but no time change, and (2) with time change but no gauge. After Theorem 1 we will be able to exhibit a host of new examples.

Methodological notes

The reasoning above, specially the early part leading to $F(t)$ to be constant, is almost embarrassingly simple, because we have purified it from some assumptions that usually overload and obscure it. We propose that those assumptions be made separately, when we set out to turn the *theorem* into a usable *strategy*.

The two conditions (5) and (7), that together lead to a constant of motion, are quite different and independent of each other. The former only follows from a property of $q(t)$ (namely, Lagrange equations) and not at all from any special form of the family q_ε. Instead, the infinitesimal invariance Equation (7) usually requires a clever choice of the family q_ε, but it often has little or nothing to do with the Lagrange equations. Also, nowhere in this paper is there a need that $t \mapsto q_\varepsilon(t)$ be a solution to Lagrange equation when $\varepsilon \neq 0$, although of course it does not hurt, and it may be important for other purposes.

First of all, we usually "have $q(t)$" not in the sense that we know a formula for it: we will rather leave it as a symbol to which we may attach some implicit assumption, on a when-needed basis. The most common assumption will of course be that $q(t)$ solves the Lagrange equations, because the first integral ultimately requires it. However, we will see that the calculations for the infinitesimal invariance are often more natural with other conditions, that either imply or are implied by the Lagrange equations. We may call them *permissible* assumptions on $q(t)$.

By definition, a mechanical first integral is meant to be a known function of $t, q(t), \dot{q}(t), \ddot{q}(t)\ldots$ By looking at the possible formulas for the constant quantities (5), (11) and the most general (48), what we seem to need is that the three subexpressions

$$\partial_\varepsilon q_\varepsilon(t)|_{\varepsilon=0}\,, \qquad \partial_\varepsilon \tau(\varepsilon,t)|_{\varepsilon=0}\,, \qquad \partial_\varepsilon G(\varepsilon,t)|_{\varepsilon=0} \tag{20}$$

be known functions of $t, q(t), \dot{q}(t), \ddot{q}(t)\ldots$, because the rest of formulas already is. To ensure this, when we set out testing candidate q_ε, τ, G for infinitesimal invariance, it is sufficient to restrict us to functions of $\varepsilon, t, q(t), q(t+\varepsilon), \dot{q}(t)$. In all examples that we have seen, q_ε and τ are indeed of that form. One contribution of this paper is a more liberal use of the $q(t+\varepsilon)$ term in building q_ε.

The situation for the gauge term G is somewhat different. We do have one example of a G which is an integral functional, and definitely *not* a function of $\varepsilon, t, q(t), q(t+\varepsilon), \dot{q}(t)\ldots$. Its special form is needed to get *finite* invariance, and not merely infinitesimal invariance (Example 1 in Section 4).

If we are content with *infinitesimal* invariance, a gauge term G which is a function of $\varepsilon, t, q(t), \dot{q}(t)$ turns up to be enough in all our examples. We have found a role for the term $q(t+\varepsilon)$ only in an alternative gauge at the end of Section 8. Actually, all of our examples for infinitesimal invariance can be worked out with a $G(\varepsilon, t)$ of the even more special form $\varepsilon \cdot \gamma(t, q(t), \dot{q}(t))$ for a known function γ.

Our calculations in the examples will follow this pattern: we propose a formula for q_ε and τ, make permissible assumptions on q_ε as we see fit, and compute the quantity

$$\left(\frac{\partial}{\partial\varepsilon}\Big(L\big(\xi, q_\varepsilon(\xi), \dot q_\varepsilon(\xi)\big)\Big|_{\xi=\tau(\varepsilon,t)} \partial_t\tau(\varepsilon,t)\Big)\right)\Big|_{\varepsilon=0} . \tag{21}$$

We need this expression to be equal to $-\partial^2_{\varepsilon,t}G(\varepsilon,t)|_{\varepsilon=0} = -\partial_t\gamma(t, q(t), \dot q(t)) = -\dot\gamma$. In other words, we need the quantity in (21) to be (the opposite of) a total time derivative of some $\gamma(t, q, \dot q)$.

In most, but not all, of our examples, the time change is missing, and the function γ will turn out to depend only on t and q. This is where we reconnect to the motivation that Levy-Leblond [7] gives when he introduces the gauge term in Noether's theorem: if we replace the Lagrangian L with $L_\gamma := L + \dot\gamma$ (what is usually called a *"gauge transform"*), then the new Lagrangian has the same Lagrange equations, and the infinitesimal invariance (10) can be written as

$$\Big(\frac{\partial}{\partial\varepsilon}L_\gamma\big(t, q_\varepsilon(t), \dot q_\varepsilon(t)\big)\Big)\Big|_{\varepsilon=0} = 0\,, \tag{22}$$

which has the simpler form of equation (8). In other terms, if the original Lagrangian is not infinitesimally invariant under the space change q_ε, then try with a gauge-transformed Lagrangian.

In Sections 7 and 8 our gauge terms depend on $\dot q$, or even on $q(t+\varepsilon)$, and so they do not lend themselves directly to the gauge-transform interpretation.

3 Noether's theorem

Theorem 1 (Equivalent invariance conditions)**.** *Let $L\colon \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n \to \mathbb{R}$ be a C^2 Lagrangian. Let I be an interval, and $(\varepsilon,t)\mapsto q_\varepsilon(t)$ be a C^2 mapping from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}^N$. Moreover, let $\tau, G\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be C^2 functions such that*

$$\tau(0,t) \equiv t \qquad \forall t\,. \tag{23}$$

Define the additional gauge $\mathcal{G}$ and time change $\mathcal{T}$:

$$\mathcal{G}(\varepsilon,t) := \varepsilon\cdot L\big(t, q_0(t), \dot q_0(t)\big)\big(\partial_\varepsilon\tau(\varepsilon,t)|_{\varepsilon=0}\big)\,, \tag{24}$$

$$\mathcal{T}(\varepsilon,t) := \varepsilon\cdot\frac{\big(\partial_\varepsilon G(\varepsilon,t)|_{\varepsilon=0}\big)}{L\big(t, q_0(t), \dot q_0(t)\big)}\,. \tag{25}$$

(If needed, we will restrict the time t to an interval where the denominator of T does not vanish). Then the following three conditions are equivalent:

1. *the infinitesimal gauge-invariance of formula* (19) *holds with τ replaced by the time change $\tau + T$ and the gauge G replaced by the trivial constant* 0*, i.e.:*

$$\frac{\partial}{\partial\varepsilon}\Big(L\big(\xi, q_\varepsilon(\xi), \dot q_\varepsilon(\xi)\big)\big|_{\xi=\tau(\varepsilon,t)+\mathcal{T}(\varepsilon,t)}\big(\partial_t\tau(\varepsilon,t)+\partial_t\mathcal{T}(\varepsilon,t)\big)\Big)\Big|_{\varepsilon=0} \equiv 0\,. \tag{26}$$

2. *the infinitesimal gauge-invariance of formula* (19) *holds with* τ *replaced by the trivial time change* $(\varepsilon, t) \mapsto t$ *and the gauge* G *replaced by* $G + \mathcal{G}$*, i.e.,*

$$\frac{\partial}{\partial \varepsilon}\Big(L\big(t, q_\varepsilon(t), \dot{q}_\varepsilon(t)\big) + \partial_t G(\varepsilon, t) + \partial_t \mathcal{G}(\varepsilon, t)\Big)\Big|_{\varepsilon=0} \equiv 0\,; \tag{27}$$

3. *the infinitesimal gauge-invariance of formula* (19) *holds for the given time change* τ *and gauge* G*, i.e.,*

$$\frac{\partial}{\partial \varepsilon}\Big(L\big(\xi, q_\varepsilon(\xi), \dot{q}_\varepsilon(\xi)\big)\Big|_{\xi=\tau(\varepsilon,t)} \partial_t \tau(\varepsilon, t) + \partial_t G(\varepsilon, t)\Big)\Big|_{\varepsilon=0} \equiv 0\,. \tag{28}$$

Proof. It is convenient to set

$$\mathcal{L}(\varepsilon, \xi) := L\big(\xi, q_\varepsilon(\xi), \dot{q}_\varepsilon(\xi)\big)\,. \tag{29}$$

In terms of $\mathcal{L}$, the left-hand sides of equations (26), (27) and (28) are respectively

$$\nu_1(t) := \frac{\partial}{\partial \varepsilon}\Big(\mathcal{L}\big(\varepsilon, \tau(\varepsilon, t) + \mathcal{T}(\varepsilon, t)\big)\partial_t\big(\tau(\varepsilon, t) + \mathcal{T}(\varepsilon, t)\big)\Big)\Big|_{\varepsilon=0}, \tag{30}$$

$$\nu_2(t) := \frac{\partial}{\partial \varepsilon}\Big(\mathcal{L}(\varepsilon, t) + \partial_t G(\varepsilon, t) + \partial_t \mathcal{G}(\varepsilon, t)\Big)\Big|_{\varepsilon=0}, \tag{31}$$

$$\nu_3(t) := \frac{\partial}{\partial \varepsilon}\Big(\mathcal{L}\big(\varepsilon, \tau(\varepsilon, t)\big)\partial_t \tau(\varepsilon, t) + \partial_t G(\varepsilon, t)\Big)\Big|_{\varepsilon=0}. \tag{32}$$

We claim that these expressions are identically the same. Their common value can be expanded out in terms of $\mathcal{L}, \tau, G$ as:

$$\mathcal{L}(0, t)\partial^2_{\varepsilon,t}\tau(0, t) + \partial_t \mathcal{L}(0, t)\partial_\varepsilon \tau(0, t) + \partial_\varepsilon \mathcal{L}(0, t) + \partial^2_{\varepsilon,t} G(0, t)\,. \tag{33}$$

This is a straightforward brute-force computation using basic two-variable chain rule calculus, with a little care due to nesting, and using the simplification rules

$$\tau(0, t) \equiv t\,, \qquad \partial_t \tau(0, t) \equiv 1\,. \tag{34}$$

However, a more meaningful proof employs the following three integral functions of the parameter ε:

$$\begin{aligned} f_{1,a,b}(\varepsilon) &:= A_{(\tau+\mathcal{T})(\varepsilon,a),(\tau+\mathcal{T})(\varepsilon,b)}(q_\varepsilon) = \\ &= \int_{(\tau+\mathcal{T})(\varepsilon,a)}^{(\tau+\mathcal{T})(\varepsilon,b)} L\big(\varepsilon, q_\varepsilon(\xi), \dot{q}_\varepsilon(\xi)\big)\, d\xi = \\ &= \int_{(\tau+\mathcal{T})(\varepsilon,a)}^{(\tau+\mathcal{T})(\varepsilon,b)} \mathcal{L}(\varepsilon, \xi)\, d\xi = && (35) \\ &= \int_a^b \mathcal{L}\big(\varepsilon, (\tau + \mathcal{T})(\varepsilon, t)\big)\partial_t(\tau + \mathcal{T})(\varepsilon, t)\, dt\,, && (36) \end{aligned}$$

$$
\begin{aligned}
f_{2,a,b}(\varepsilon) &:= A_{a,b}(q_\varepsilon) + G(\varepsilon,b) + H(\varepsilon,b) - G(\varepsilon,a) - H(\varepsilon,a) = \\
&= \int_a^b \mathcal{L}(\varepsilon,t)\,dt + G(\varepsilon,b) + \mathcal{G}(\varepsilon,b) - G(\varepsilon,a) - \mathcal{G}(\varepsilon,a)\,, \qquad (37)\\
f_{3,a,b}(\varepsilon) &:= A_{\tau(\varepsilon,a),\tau(\varepsilon,b)}(q_\varepsilon) + G(\varepsilon,b) - G(\varepsilon,a) = \\
&= \int_{\tau(\varepsilon,a)}^{\tau(\varepsilon,b)} \mathcal{L}(\varepsilon,\xi)\,d\xi + G(\varepsilon,b) - G(\varepsilon,a) = \qquad (38)\\
&= \int_a^b \mathcal{L}\big(\varepsilon,\tau(\varepsilon,t)\big)\partial_t\tau(\varepsilon,t)\,d\xi + G(\varepsilon,b) - G(\varepsilon,a)\,. \qquad (39)
\end{aligned}
$$

By derivating formulas (36), (37) and (39), whose integrals have fixed endpoints, it is clear that

$$
f'_{i,a,b}(0) = \int_a^b \nu_i(\xi)d\xi \qquad \text{for } i = 1,2,3. \qquad (40)
$$

Let us now recalculate $f'_{1,a,b}(0)$ by differentiating the alternative formula (35) with respect to ε under the integral sign:

$$
\begin{aligned}
f'_{1,a,b}(\varepsilon) = &\int_{(\tau+\mathcal{T})(\varepsilon,a)}^{(\tau+\mathcal{T})(\varepsilon,b)} \partial_\varepsilon\mathcal{L}(\varepsilon,\xi)\,d\xi + \\
&+ \mathcal{L}\big(\varepsilon,(\tau+\mathcal{T})(\varepsilon,\xi)\big)\partial_\varepsilon(\tau+\mathcal{T})(\varepsilon,\xi)\Big|_{\xi=b} - \qquad (41)\\
&- \mathcal{L}\big(\varepsilon,(\tau+\mathcal{T})(\varepsilon,\xi)\big)\partial_\varepsilon(\tau+\mathcal{T})(\varepsilon,\xi)\Big|_{\xi=a}\,. \qquad (42)
\end{aligned}
$$

From equations (25) and (34) we obtain

$$
\begin{aligned}
(\tau+\mathcal{T})(\varepsilon,\xi)\big|_{\varepsilon=0} &= \tau(0,\xi) + \mathcal{T}(0,\xi) = \xi\,,\\
\partial_\varepsilon(\tau+\mathcal{T})(\varepsilon,\xi)\big|_{\varepsilon=0} &= \partial_\varepsilon\tau(0,\xi) + \frac{\partial_\varepsilon G(0,\xi)}{\mathcal{L}(0,\xi)}\,,
\end{aligned}
$$

so that, when $\varepsilon = 0$, the common form of the two terms (41) and (42) rewrites in terms of $\mathcal{L},\tau,G$ this way:

$$
\begin{aligned}
&\mathcal{L}(0,(\tau+\mathcal{T})(0,\xi))\partial_\varepsilon(\tau+\mathcal{T})(0,\xi) = \\
&\quad = \mathcal{L}(0,\xi)\partial_\varepsilon\tau(0,\xi) + \mathcal{L}(0,\xi)\cdot\frac{\partial_\varepsilon G(0,t)}{\mathcal{L}(0,\xi)} = \mathcal{L}(0,\xi)\partial_\varepsilon\tau(0,\xi) + \partial_\varepsilon G(0,\xi)\,.
\end{aligned}
$$

Hence the following additional form for $f'_{1,a,b}(0)$:

$$
\begin{aligned}
f'_{1,a,b}(0) = &\int_a^b \partial_\varepsilon\mathcal{L}(\varepsilon,\xi)\big|_{\varepsilon=0}\,d\xi + \qquad (43)\\
&+ \Big(\mathcal{L}(0,\xi)\partial_\varepsilon\tau(0,\xi) + \partial_\varepsilon G(0,\xi)\Big)\Big|_{\xi=b} - \qquad (44)\\
&- \Big(\mathcal{L}(0,\xi)\partial_\varepsilon\tau(0,\xi) + \partial_\varepsilon G(0,\xi)\Big)\Big|_{\xi=a}\,. \qquad (45)
\end{aligned}
$$

At exactly this same right-hand side, but with much less effort, the readers will end up if they recalculate $f'_{2,a,b}(0)$ and $f'_{3,a,b}(0)$ by differentiating the alternative formulas (37) and (38) respectively. In particular,

$$f'_{1,a,b}(0) = f'_{2,a,b}(0) = f'_{3,a,b}(0) \qquad \forall a, b\,.$$

As announced, we deduce that $\nu_1 \equiv \nu_2 \equiv \nu_3$ and that Conditions 1, 2 and 3 are indeed equivalent to each other and to

$$f'_{1,a,b}(0) = 0 \qquad \forall a, b. \tag{46}$$

□

Observation 1. Formula (25) is not the only possible auxiliary time change that makes the theorem work. An alternative choice is

$$\tilde{\mathcal{T}}(\varepsilon, t) := \frac{G(\varepsilon, t)}{L\big(t, q_0(t), \dot{q}_0(t)\big)}, \tag{47}$$

which is possibly nonlinear in ε.

Theorem 2 (Noether's theorem)**.** *Given L, q, τ, G as in Theorem 1, suppose that the three equivalent conditions hold, and that also $t \mapsto q_0(t)$ is a solution to the Lagrange equation. Then the following function is constant:*

$$\begin{aligned} N(t) := {} & \partial_{\dot{q}} L\big(t, q_0(t), \dot{q}_0(t)\big) \cdot \partial_\varepsilon q_\varepsilon(t)|_{\varepsilon=0} + \\ & + L\big(t, q_0(t), \dot{q}_0(t)\big) \partial_\varepsilon \tau(\varepsilon, t)|_{\varepsilon=0} + \\ & + \partial_\varepsilon G(\varepsilon, t)|_{\varepsilon=0}\,. \end{aligned} \tag{48}$$

Observation 2. One can check that the value of the constant of motion in equation (48) does not change if we perform either the replacements $\tau \to \tau + \mathcal{T}$, $G \to 0$ of Condition 1, or the replacements $\tau \to t$, $G \to H + \mathcal{G}$ of Condition 2.

Proof. We continue where we left off with the proof of Theorem 1. We already detect a piece of $N(t)$ in formulas (44) and (45). Let us turn our attention to the integral term (43). Using the chain rule in the original Lagrangian and reversing the order of the mixed derivatives $\partial^2_{\varepsilon,\xi}$, the expression $\partial_\varepsilon \mathcal{L}(\varepsilon, \xi)$ can be expanded out as

$$\partial_\varepsilon \mathcal{L}(\varepsilon, \xi) = \partial_q L\big(\xi, q_\varepsilon(\xi), \dot{q}_\varepsilon(\xi)\big) \cdot \partial_\varepsilon q_\varepsilon(\xi) + \partial_{\dot{q}} L\big(\xi, q_\varepsilon(\xi), \dot{q}_\varepsilon(\xi)\big) \cdot \frac{\partial}{\partial \xi} \partial_\varepsilon q_\varepsilon(\xi)\,.$$

From now on, assume that q_0 is a solution to the Lagrange equations (2). Then

$$\begin{aligned} \partial_\varepsilon \mathcal{L}(\varepsilon, \xi)\big|_{\varepsilon=0} = {} & \Big(\frac{d}{d\xi} \partial_{\dot{q}} L\big(\xi, q_0(\xi), \dot{q}_0(\xi)\big)\Big) \cdot \partial_\varepsilon q_\varepsilon(\xi)\big|_{\varepsilon=0} + \\ & + \partial_{\dot{q}} L\big(\xi, q_0(\xi), \dot{q}_0(\xi)\big) \cdot \frac{d}{d\xi}\big(\partial_\varepsilon q_\varepsilon(\xi)\big|_{\varepsilon=0}\big) = \end{aligned}$$

$$= \frac{d}{d\xi}\Big(\partial_{\dot q} L\big(\xi, q_0(\xi), \dot q_0(\xi)\big) \cdot \partial_\varepsilon q_\varepsilon(\xi)\big|_{\varepsilon=0}\Big)\,.$$

The fundamental theorem of calculus now exposes the remaining piece of $N(t)$ in the integral of formula (43):

$$\begin{aligned}\int_a^b \partial_\varepsilon \mathcal{L}(\varepsilon,\xi)\big|_{\varepsilon=0}\, d\xi = \int_a^b \frac{d}{d\xi}\Big(\partial_{\dot q} L\big(\xi, q_0(\xi), \dot q_0(\xi)\big) \cdot \big(\partial_\varepsilon q_\varepsilon(\xi)\big|_{\varepsilon=0}\big)\Big) d\xi = \\ = \partial_{\dot q} L\big(\xi, q_0(\xi), \dot q_0(\xi)\big) \cdot \partial_\varepsilon q_\varepsilon(\xi)\Big|_{\varepsilon=0,\xi=b} - \\ - \partial_{\dot q} L\big(\xi, q_0(\xi), \dot q_0(\xi)\big) \cdot \partial_\varepsilon q_\varepsilon(\xi)\Big|_{\varepsilon=0,\xi=a}\,.\end{aligned}$$

This leads to one last expression for $f'_{1,a,b}(0)$:

$$f'_{1,a,b}(0) = N(b) - N(a)\,. \tag{49}$$

which, combined with the null derivative assumption (46), implies that $N(b) = N(a)$. Since a, b are arbitrary, we conclude that N is constant whenever q_0 solves the Lagrange equations and the equivalent Conditions 1, 2 and 3 hold. □

Observation 3. We may feel uneasy that formula (25) for $\mathcal{T}$ contains the Lagrangian L at the denominator:

$$\mathcal{T}(\varepsilon, t) := \varepsilon \frac{\partial_\varepsilon G(0,t)}{L\big(t, q_0(t), \dot q_0(t)\big)}\,. \tag{50}$$

What happens if the Lagrangian vanishes for some values of t? Are those values of any intrinsic importance in the no-gauge approach? Luckily the answer is negative, thanks to this simple trick: choose a constant k so that $L(t, q_0(t), \dot q_0(t)) + k$ does not vanish in a compact interval we are interested in, and define the modified functions:

$$L_k = L + k\,, \qquad \mathcal{T}_k(\varepsilon, t) := \varepsilon \frac{\partial_\varepsilon G(0,t) - k\partial_\varepsilon \tau(0,t)}{L\big(t, q_0(t), \dot q_0(t)\big) + k}\,. \tag{51}$$

Then we may substitute the following Condition 4 for Condition 1 in Theorem 1:

4. the infinitesimal gauge-invariance of formula (19) holds with L replaced by L_k, the time change τ replaced by $\tau + \mathcal{T}_k$ and the gauge G replaced by the trivial constant 0, i.e.:

$$\frac{\partial}{\partial\varepsilon}\Big(L_k\big(\xi, q_\varepsilon(\xi), \dot q_\varepsilon(\xi)\big)\big|_{\xi=\tau(\varepsilon,t)+\mathcal{T}_k(\varepsilon,t)}\big(\partial_t \tau(\varepsilon,t) + \partial_t \mathcal{T}_k(\varepsilon,t)\big)\Big)\Big|_{\varepsilon=0} \equiv 0\,. \tag{52}$$

Of course the Lagrangians L and L_k have the same Lagrange equations. Also, the value of the first integral in equation (48) does not change if L is replaced by L_k, and τ by $\tau + \mathcal{T}_k$.

4 Energy, momentum, angular momentum

Example 1. Let us see how Theorem 1 works out for Conservation of Energy when the Lagrangian $L(q,\dot q)$ is autonomous. As already noted in Section 2, there is infinitesimal invariance of Condition 3 type with the choices

$$q_\varepsilon(t) := q(t+\varepsilon)\,, \qquad \tau_1(\varepsilon,t) := t\,, \qquad G_1(\varepsilon,t) := -\varepsilon L\big(q(t),\dot q(t)\big)\,. \tag{53}$$

Formulas (24) and (25) then become

$$\mathcal{G}_1(\varepsilon,t) := 0\,, \qquad \mathcal{T}_1(\varepsilon,t) := \varepsilon\frac{\partial_\varepsilon G(0,t)}{L\big(q(t),\dot q(t)\big)} = \varepsilon\frac{-L\big(q(t),\dot q(t)\big)}{L\big(q(t),\dot q(t)\big)} = -\varepsilon\,, \tag{54}$$

Theorem 1 says that there is infinitesimal invariance also of Condition 1 type, that is, with the alternative choices

$$q_\varepsilon(t) := q(t+\varepsilon)\,, \quad \tau_2(\varepsilon,t) := \tau_1(\varepsilon,t) + \mathcal{T}_1(\varepsilon,t) = t-\varepsilon\,, \quad G_2(\varepsilon,t) := 0\,. \tag{55}$$

If we had started out with these last choices (55), we would get

$$\mathcal{G}_2(\varepsilon,t) := \varepsilon L\big(q(t),\dot q(t)\big)\big(\partial_\varepsilon\tau_2(\varepsilon,t)\big|_{\varepsilon=0}\big) = -\varepsilon L\big(q(t),\dot q(t)\big)\,, \quad \mathcal{T}_2 := 0\,, \tag{56}$$

and we would be led back to infinitesimal invariance of Condition 2 type with the original choice (53). Whichever the approach, in the end the energy first integral of equation (48) becomes

$$\partial_{\dot q}L(q,\dot q)\cdot\dot q - L(q,\dot q)\,. \tag{57}$$

It is unpleasant that the invariance with nontrivial time change given by (55) is actually a finite invariance, while the invariance with gauge given by (53) is merely infinitesimal. We propose here the following alternative gauge choice

$$q_\varepsilon(t) := q(t+\varepsilon)\,, \quad \tau_2(\varepsilon,t) := t\,, \quad G_2(\varepsilon,t) := -\int_t^{t+\varepsilon} L\big(q(\xi),\dot q(\xi)\big)d\xi \tag{58}$$

which recovers a perfect finite invariance. In fact, the quantity

$$\begin{aligned}
f_{a,b}(\varepsilon) &:= A_{\tau_2(\varepsilon,a),\tau_2(\varepsilon,b)}(q_\varepsilon) + G_2(\varepsilon,b) - G_2(\varepsilon,a) = \\
&= \int_{\tau(\varepsilon,a)}^{\tau(\varepsilon,b)} L\big(q_\varepsilon(\xi),\dot q_\varepsilon(\xi)\big)\,d\xi + G_2(\varepsilon,b) - G_2(\varepsilon,a) = \\
&= \int_a^b L\big(q(\xi+\varepsilon),\dot q(\xi+\varepsilon)\big)\,d\xi + G_2(\varepsilon,b) - G_2(\varepsilon,a) = \\
&= \int_{a+\varepsilon}^{b+\varepsilon} L\big(q(t),\dot q(t)\big)\,dt + G_2(\varepsilon,b) - G_2(\varepsilon,a) = \\
&= \int_{a+\varepsilon}^{b+\varepsilon} L\big(q(t),\dot q(t)\big)\,dt - \int_b^{b+\varepsilon} L\big(q(\xi),\dot q(\xi)\big)d\xi +
\end{aligned} \tag{59}$$

$$+\int_a^{a+\varepsilon} L\big(q(\xi),\dot q(\xi)\big)d\xi = \tag{60}$$

$$= \int_a^b L\big(q(t),\dot q(t)\big)\,dt\,, \tag{61}$$

does not depend on ε. Or again, if your favourite mnemonic reference is the non-integrated formula (19), the expression

$$\begin{aligned} L\big(q_\varepsilon(\xi),\dot q_\varepsilon(\xi)\big)\Big|_{\xi=\tau_1(\varepsilon,t)} &\partial_t\tau_1(\varepsilon,t) + \partial_t G_2(\varepsilon,t) = \\ &= L\big(q(t+\varepsilon),\dot q(t+\varepsilon)\big) - \Big(L\big(q(t+\varepsilon),\dot q(t+\varepsilon)\big) - L\big(q(t),\dot q(t)\big)\Big) \\ &= L\big(q(t),\dot q(t)\big) \end{aligned} \tag{62}$$

does not depend on ε. The associated first integral is again the energy. The gauge choice (58) seems to be unusual, because it is an *integral functional*, and not a point function of $q(t),\dot q(t)$.

We could apply again Theorem 1 to the triple q_ε,τ_2,G_2 and get a new variant with trivial gauge and nontrivial time change, but unfortunately we have to relinquish the finiteness of the invariance.

Example 2. Suppose that $L(t,q,\dot q)$ is invariant in the direction of $u\in\mathbb{R}^n$:

$$L(t,q+\varepsilon u,\dot q) \equiv L(t,q,\dot q) \qquad \forall t,\varepsilon\in\mathbb{R},\quad q,\dot q\in\mathbb{R}^n\,. \tag{63}$$

Then for any $q(t)$ there is obvious finite invariance for the translated family

$$q_\varepsilon(t) := q(t)+\varepsilon u, \qquad \tau(\varepsilon,t)\equiv t, \qquad G\equiv 0\,. \tag{64}$$

The constant of motion when $q(t)$ solves the Lagrange equations is the component of the momentum in the direction of u:

$$\partial_{\dot q} L\big(t,q(t),\dot q(t)\big)\cdot u\,. \tag{65}$$

Of course here $\mathcal{G}=\mathcal{T}\equiv 0$, and all three conditions of the theorem literally collapse into one.

Example 3. Consider a point in the $\mathbb{R}^2$ plane that is driven by a (possibly time-dependent) central force field:

$$L(t,q,\dot q) := \frac{1}{2}\|\dot q\|^2 + V\big(t,\|q\|\big)\,. \tag{66}$$

Given a smooth trajectory $q(t)$ in $\mathbb{R}^2$, define the rotation family

$$q_\varepsilon(t) := \begin{pmatrix}\cos\varepsilon & -\sin\varepsilon\\ \sin\varepsilon & \cos\varepsilon\end{pmatrix} q(t)\,, \qquad \tau(\varepsilon,t) := t,, \qquad G(\varepsilon,t) := 0\,. \tag{67}$$

It is clear that we have finite invariance:

$$\frac{\partial}{\partial\varepsilon} L\big(t,q_\varepsilon(t),\dot q_\varepsilon(t)\big) = 0\,. \tag{68}$$

Noether's theorem gives the first integral of angular momentum for all Lagrange motions

$$\partial_{\dot q} L \cdot \partial_\varepsilon q_\varepsilon|_{\varepsilon=0} = \dot q \cdot \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} q = \det(q, \dot q). \tag{69}$$

Again $\mathcal{G} \equiv \mathcal{T} \equiv 0$.

5 The free fall

Consider the familiar free fall of a particle $q = (x, y, z) \in \mathbb{R}^3$, subject to its own weight, with uniform gravity acceleration g pointing in the negative z direction, and no air resistance. The Lagrange function is the difference of kinetic and potential energy:

$$L(q, \dot q) = \frac{1}{2}\|\dot q\|^2 - (0, 0, g) \cdot q = \frac{1}{2}\|\dot q\|^2 + gz\,. \tag{70}$$

The associated Lagrange equation is of course $\ddot q = (0, 0, -g)$. Noether's theorem applies to this system in various ways. One is because it is an autonomous system, so that using either (53) or (55) we obtain conservation of energy:

$$\partial_{\dot q} L(q, \dot q) \cdot \dot q - L(q, \dot q) = \dot q \cdot \dot q - L(q, \dot q) = \frac{1}{2}\|\dot q\|^2 + gz\,. \tag{71}$$

Another obvious invariance is with respect to horizontal translations: if we choose $u = (u_1, u_2, 0)$ we can use the translation family (64) and obtain the conservation of

$$\partial_{\dot q} L(q, \dot q) \cdot u = \dot q \cdot u = u_1 \dot x + u_2 \dot y \tag{72}$$

for any $u_1, u_2 \in \mathbb{R}$, whence $\dot x, \dot y$ are constant.

A third invariance is less obvious. Let us try with the vertical translation family

$$q_\varepsilon(t) := q(t) + (0, 0, \varepsilon)\,. \tag{73}$$

The Lagrangian is not itself invariant, but we can write

$$\frac{\partial}{\partial \varepsilon} L\big(q_\varepsilon, \dot q_\varepsilon\big) = g = \partial^2_{\varepsilon,t}(\varepsilon g t)\,, \tag{74}$$

which suggests that to get (finite) invariance it is enough to introduce a non-trivial gauge term:

$$G(\varepsilon, t) := -\varepsilon g t, \qquad \tau(\varepsilon, t) = t\,. \tag{75}$$

The first integral that follows is

$$\partial_{\dot q} L\big(q(t), \dot q(t)\big) \cdot \partial_\varepsilon q_\varepsilon(t)|_{\varepsilon=0} + \partial_\varepsilon G(\varepsilon, t)|_{\varepsilon=0} = \dot z - gt, \tag{76}$$

which can be combined with the other integrals $\dot x, \dot y$ into the vector first integral

$$\dot q - (0, 0, gt)\,. \tag{77}$$

Following Theorem 1 we can define

$$\mathcal{T}(\varepsilon,t) := \varepsilon\frac{\partial_\varepsilon G(0,t)}{L\big(q(t),\dot q(t)\big)} = -\varepsilon\frac{gt}{L\big(q(t),\dot q(t)\big)}\,. \tag{78}$$

and obtain infinitesimal invariance with the family (73), time change $(\varepsilon,t)\mapsto t+\mathcal{T}(\varepsilon,t)$ and null gauge. We can check this fact directly with a simple calculation:

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\Big(L\big(q_\varepsilon(t-\varepsilon gt/L),\dot q_\varepsilon(t-\varepsilon gt/L)\big)\big(1-\varepsilon g/L+\varepsilon gt\dot L/L^2\big)\Big)\Big|_{\varepsilon=0} &=\\ = g+\dot L(-gt/L)+L\big(-g/L+gt\dot L/L^2\big)\equiv 0\,.&\end{aligned} \tag{79}$$

Also, the conserved function (76) can be obtained from the general formula (48) with the substitutions $G\to 0$ and $\tau\to t+\mathcal{T}$:

$$\begin{aligned}\partial_{\dot q}L\big(q(t),\dot q(t)\big)\cdot\partial_\varepsilon q_\varepsilon(t)|_{\varepsilon=0}+L\big(q(t),\dot q(t)\big)\partial_\varepsilon\tau(\varepsilon,t)|_{\varepsilon=0} &=\\ = \dot q\cdot(0,0,1)+L(-gt/L)=\dot z-gt\,.&\end{aligned} \tag{80}$$

The free fall system can also be described in a different way. As well known (and trivially checked), two Lagrangian functions give the same Lagrange equation if they differ by the total time derivative of a function of (t,q)

$$L_\gamma(t,q,\dot q) = L(t,q,\dot q)+\partial_t\gamma(t,q)+\partial_q\gamma(t,q)\cdot\dot q\,. \tag{81}$$

Changing L into L_γ is called a gauge transform. Let us take $\gamma(t,q)=-gtz$ and define the new Lagrangian function for the free fall

$$L_\gamma(t,q,\dot q) = L(q,\dot q)+\frac{d}{dt}(-gtz) = \frac{1}{2}\|\dot q\|^2-gt\dot z\,. \tag{82}$$

An advantage of L_γ is that it is independent of q, so that we get finite invariance under the translation family $q_\varepsilon(t):=q(t)+\varepsilon u$ for any $u\in\mathbb{R}^3$, with trivial $\tau(\varepsilon,t)\equiv t$ and $G\equiv 0$. The formula for the resulting first integral is $\partial_{\dot q}L_\gamma\cdot u = (\dot x,\dot y,\dot z-gt)\cdot u$, which leads directly to the vector first integral of formula (77).

Of course, energy conservation becomes more difficult, because the new Lagrangian is not anymore autonomous. Let us take again the time-shift family $q_\varepsilon(t):=q(t+\varepsilon)$, trivial $\tau(t,\varepsilon):=t$, and calculate

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}L_\gamma\big(t,q(t+\varepsilon),\dot q(t+\varepsilon)\big)\Big|_{\varepsilon=0} &= \frac{d}{dt}L_\gamma\big(t,q(t),\dot q(t)\big)-\partial_tL_\gamma(t,q(t),\dot q(t)) =\\ &= \frac{d}{dt}L_\gamma\big(t,q(t),\dot q(t)\big)-g\dot z(t) = \frac{d}{dt}\Big(L_\gamma\big(t,q(t),\dot q(t)\big)+gz(t)\Big)\,.\end{aligned} \tag{83}$$

This formula suggests to introduce a new gauge

$$G_1(\varepsilon,t) := -\varepsilon\big(L_\gamma(t,q(t),\dot q(t))+gz\big)\,, \tag{84}$$

which realizes infinitesimal invariance

$$\frac{\partial}{\partial\varepsilon}\Big(L_\gamma\big(t,q_\varepsilon(t),\dot q_\varepsilon(t)\big)+\partial_tG_1(\varepsilon,t)\Big)\Big|_{\varepsilon=0}\equiv 0\,. \tag{85}$$

The resulting first integral is the energy:

$$\partial_{\dot q} L_\gamma\big(t, q(t), \dot q(t)\big) \cdot \partial_\varepsilon q_\varepsilon(t)|_{\varepsilon=0} + \partial_\varepsilon G_1(\varepsilon, t)|_{\varepsilon=0} = \\ = \dot q \cdot \dot q - g t \dot z - L_\gamma - g z = \frac{1}{2}\|\dot q\|^2 - g z \,. \quad (86)$$

Again according to Theorem 1, we can define

$$\mathcal{T}_1(\varepsilon, t) := \varepsilon \frac{\partial_\varepsilon G_1(0,t)}{L_\gamma\big(t, q(t), \dot q(t)\big)} = -\varepsilon\Big(1 + \frac{g z(t)}{L_\gamma\big(t, q(t), \dot q(t)\big)}\Big)\,, \quad (87)$$

and get infinitesimal invariance with the same q_ε and the replacements $\tau \to \tau + \mathcal{T}_1$, $G_1 \to 0$, leading again to conservation of energy. The reader can check this by direct calculation.

6 Kepler's problem

In all the examples that we have seen so far, the infinitesimal invariance does not require any precondition on $q(t)$. In this Section we will see examples where we need such preconditions, and we must be careful that they are compatible with being a solution to Lagrange equations.

The Lagrangian function of Kepler's problem is

$$K\big(q, \dot q\big) = \frac{1}{2}\|\dot q\|^2 + \frac{k}{\|q\|}\,, \quad (88)$$

with its associated Lagrange equation

$$\ddot q(t) = -k\frac{q(t)}{\|q(t)\|^3}\,, \quad (89)$$

where $k > 0$ is a parameter. Notice that for all Kepler motions q is parallel to $\ddot q$. We will assume that q is a vector in $\mathbb{R}^2$ to simplify some formulas.

As for all autonomous Lagrangians, the energy

$$\partial_{\dot q} K \cdot \dot q - L = \frac{1}{2}\|\dot q\|^2 - \frac{k}{\|q\|} \quad (90)$$

is conserved. The Lagrangian is also invariant under rotations around the origin. Noether's theorem gives the first integral of angular momentum $\det(q, \dot q)$.

Kepler's system enjoys also a recondite infinitesimal invariance. Let us take again a smooth trajectory $q(t)$ in $\mathbb{R}^2 \setminus \{(0,0)\}$ (not necessarily a solution to Lagrange equations, for now), a vector $u \in \mathbb{R}^2$, and define the family

$$q_\varepsilon(t) := q(t) + \big(q(t)\cdot u\big) q(t+\varepsilon) - \big(q(t+\varepsilon)\cdot u\big) q(t) = \\ = q(t) + \det\big(q(t), q(t+\varepsilon)\big) u^\perp \quad (91)$$

with $u^\perp = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) u$. It is clear that $q_0(t) = q(t)$. You can visualize some sample trajectories in Figure 3. This family q_ε is different and simpler than the one

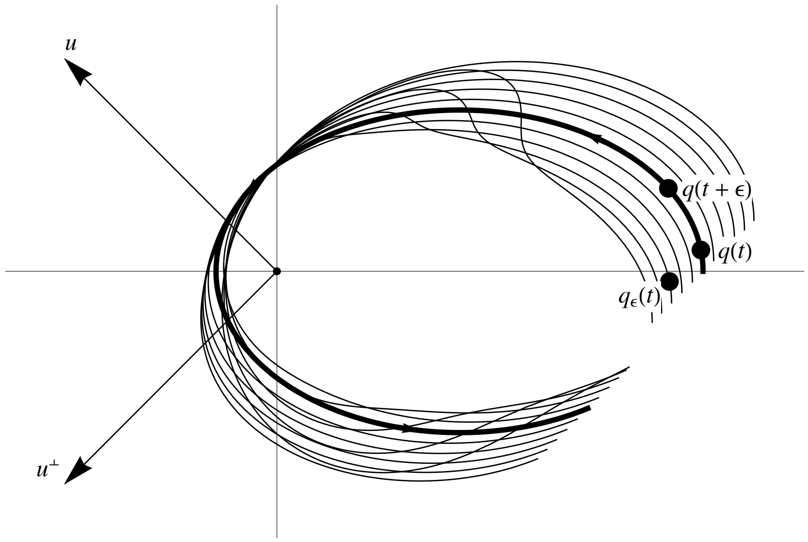

Figure 3: The q_ε family for the Laplace-Runge-Lenz vector, built around an elliptic orbit (the thick curve). For fixed t the various $q_\varepsilon(t)$ are on a straight line parallel orthogonal to u

found in the literature (for example, see Lévy-Leblond [7], formula (36)), whose formula would be

$$q(t) + \frac{\varepsilon}{2}\Big(2\big(q(t)\cdot u\big)\dot{q}(t) - \big(\dot{q}(t)\cdot u\big)q(t) - \big(\dot{q}(t)\cdot q(t)\big)u\Big). \tag{92}$$

The reader can check that the following relations hold:

$$\frac{\partial}{\partial\varepsilon}K\big(q_\varepsilon(t),\dot{q}_\varepsilon(t)\big)\Big|_{\varepsilon=0} = \tag{93}$$

$$= \det\big(q(t),\ddot{q}(t)\big)\det\big(u,\dot{q}(t)\big) - k\frac{\det\big(u,q(t)\big)\det\big(q(t),\dot{q}(t)\big)}{\big\|q(t)\big\|^3} = \tag{94}$$

$$= \det\big(q(t),\ddot{q}(t)\big)\det\big(u,\dot{q}(t)\big) + \frac{\partial}{\partial t}\left(k\frac{q(t)\cdot u}{\|q(t)\|}\right). \tag{95}$$

If we define the gauge

$$G(\varepsilon,t) := -\varepsilon k\frac{q(t)\cdot u}{\|q(t)\|}, \tag{96}$$

then

$$\frac{\partial}{\partial\varepsilon}\left(K\big(q_\varepsilon(t),\dot{q}_\varepsilon(t)\big) + \partial_t G(\varepsilon,t)\right)\Big|_{\varepsilon=0} = \det\big(q(t),\ddot{q}(t)\big)\det\big(u,\dot{q}(t)\big)\,. \tag{97}$$

If $q(t)$ is any motion for which $\ddot{q}(t)$ is parallel to $q(t)$, then we have infinitesimal invariance of the form (10):

$$\frac{\partial}{\partial\varepsilon}\Big(K\big(q_\varepsilon(t),\dot{q}_\varepsilon(t)\big)+\partial_t G(\varepsilon,t)\Big)\Big|_{\varepsilon=0}\equiv 0\,, \tag{98}$$

which is covered by Theorem 1, condition 3, with the trivial $\tau(\varepsilon,t)\equiv t$. Also, $\ddot{q}(t)$ and $q(t)$ are parallel whenever $q(t)$ is a solution to Kepler's equation (89). Therefore Noether's Theorem 2 yields the following constant of motion:

$$\begin{aligned}\partial_{\dot{q}}K\big(q(t),\dot{q}(t)\big)\cdot\partial_\varepsilon q_\varepsilon(t)|_{\varepsilon=0}+\partial_\varepsilon G(\varepsilon,t)|_{\varepsilon=0}&=\\ =(q\cdot u)\|\dot{q}\|^2-(\dot{q}\cdot u)(\dot{q}\cdot q)-k\frac{q\cdot u}{\|q\|}&.\end{aligned} \tag{99}$$

Since the vector $u\in\mathbb{R}^2$ is arbitrary, we have the vector-valued first integral

$$q\|\dot{q}\|^2-(\dot{q}\cdot q)\dot{q}-k\frac{q}{\|q\|}\,, \tag{100}$$

which is called the Laplace-Runge-Lenz vector.

According to Theorem 1, if we define the additional time change $\mathcal{T}$ as

$$\mathcal{T}(\varepsilon,t):=\varepsilon\frac{\partial_\varepsilon G(0,t)}{K\big(q(t),\dot{q}(t)\big)}=-\varepsilon k\frac{q(t)\cdot u}{\|q(t)\|K\big(q(t),\dot{q}(t)\big)}\,. \tag{101}$$

there is infinitesimal invariance also under the nontrivial time change $(\varepsilon,t)\mapsto t+\mathcal{T}\varepsilon,t)$, without gauge.

Let us show a simple example where Noether's theorem is applied to a *particular* solution, leading to a nontrivial function that is constant along that single solution, but not along most others. Consider the following family of uniform circular motions in the plane with the same period but a phase shift:

$$q_\varepsilon(t):=e^\varepsilon\big(\cos(\sqrt{k}(\varepsilon+t)),\sin(\sqrt{k}(\varepsilon+t))\big) \tag{102}$$

The function $t\mapsto q_\varepsilon(t)$ is a Kepler motion only when $\varepsilon=0$, as we check at once. Still, there is infinitesimal invariance with trivial $\tau(\varepsilon,t)\equiv 0$, $G\equiv 0$:

$$\frac{\partial}{\partial\varepsilon}K\big(q_\varepsilon(t),\dot{q}_\varepsilon(t)\big)\Big|_{\varepsilon=0}=\frac{\partial}{\partial\varepsilon}\left(\frac{k}{2}e^{2\varepsilon}+\frac{k}{e^\varepsilon}\right)\Big|_{\varepsilon=0}=0 \tag{103}$$

If we apply Noether's theorem we obtain that the (square of) the speed $t\mapsto\|\dot{q}_0(t)\|^2$ is constant along the circular motion. The speed is clearly a nontrivial function that is not constant along any Kepler motions except circular ones.

Let us generalize the Kepler Lagrangian this way:

$$K_\alpha\big(q,\dot{q}\big)=\frac{1}{2}\|\dot{q}\|^\alpha+\frac{k}{\|q\|}\,, \tag{104}$$

for a real exponent α. If we carry out the computations with the same family q_ε and G as in formulas (91) and (96), we get

$$\frac{\partial}{\partial\varepsilon}\Big(K_\alpha\big(q_\varepsilon(t),\dot q_\varepsilon(t)\big)+\partial_t G(\varepsilon,t)\Big)\Big|_{\varepsilon=0}=$$

$$=\frac{\alpha}{2}\|\dot q(t)\|^{\alpha-2}\det\big(q(t),\ddot q(t)\big)\det\big(u,\dot q(t)\big)\,. \quad (105)$$

As before, we have infinitesimal invariance whenever $\ddot q(t)$ and $q(t)$ are parallel for all t. Unfortunately, this condition is not generically satisfied by solutions to the new Lagrange equations. One odd exception is the degenerate case $\alpha = 1$, when all solutions to Lagrange equations are uniform circular motions.

7 Some superintegrable systems

A recent paper [9] introduced the following Lagrangian systems in two dimensions:

$$L(x,y,\dot x,\dot y)=\dot x\dot y-g(x)y, \quad (106)$$

which are interesting because for suitable choices of the function g there is either weak Lyapunov instability or isochronicity. What matters here is that those isochronous cases exhibit the rare property of being super-integrable, because they have three independent first integrals. Now we are going to investigate the matter in terms of Noether's theorem.

Two first integrals are obvious, and they do not need any assumption on g:

$$\frac{\dot x}{2}+V(x)\,,\qquad \dot y\dot x+g(x)y \quad (107)$$

where V is any primitive of g. Notice that the Lagrange equations are

$$\ddot x=-g(x)\,,\qquad \ddot y=-g'(x)y\,. \quad (108)$$

Let us search for a third constant of motion starting from the following space-change family:

$$q_\varepsilon(t):=\big(x_\varepsilon(t),y_\varepsilon(t)\big)=\Big(x(t)+f\big(x(t)\big)y(t+\varepsilon)-f\big(x(t+\varepsilon)\big)y(t),y(t)\Big) \quad (109)$$

where f is a function to be determined. We can compute, using also the Lagrange equations (108):

$$\frac{\partial}{\partial\varepsilon}\Big(L\big(x_\varepsilon(t),y_\varepsilon(t),\dot x_\varepsilon(t),\dot y_\varepsilon(t)\big)\Big|_{\varepsilon=0}= \quad (110)$$

$$=y(t)^2\Big(x'(t)f'(x(t))g'(x(t))\Big)+ \quad (111)$$

$$+\,2y(t)y'(t)\Big(\frac{1}{2}g(x(t))f'(x(t))- \quad (112)$$

$$-\,f(x(t))g'(x(t))-\frac{1}{2}x'(t)^2f''(x(t))\Big)\Big)\,. \quad (113)$$

To make the previous expression into a total time derivative we impose that

$$\frac{d}{dt}\Big(x'(t)f'(x(t))g'(x(t))\Big) =$$
$$= \frac{1}{2}g(x(t))f'(x(t)) - f(x(t))g'(x(t)) - \frac{1}{2}x'(t)^2 f''(x(t))\Big) \,. \quad (114)$$

Expanding out the derivative on the left hand side and using again the Lagrange equations (108), equation (114) becomes

$$x'(t)\Big(f'''(x)x'(t)^2 - 3g(x)f''(x) + 3f'(x)g'(x) + 2f(x)g''(x)\Big) = 0\,. \quad (115)$$

Let us now assume that f is a polynomial of degree ≤ 2, so that the third derivative f''' vanishes. Then equation (115) simplifies to

$$-3g(x)f''(x) + 3f'(x)g'(x) + 2f(x)g''(x) = 0\,. \quad (116)$$

This is a second order linear differential equation in g with polynomial coefficients. If we assume that g solves equation (116), we can take

$$G(\varepsilon, t) = -\varepsilon y(t)^2\Big(\frac{1}{2}g(x(t))f'(x(t)) - f(x(t))g'(x(t)) - \frac{1}{2}\dot{x}(t)^2 f''(x(t))\Big)\,, \quad (117)$$

and obtain the desired infinitesimal invariance:

$$\frac{\partial}{\partial\varepsilon}\Big(L\big(x_\varepsilon(t), y_\varepsilon(t), \dot{x}_\varepsilon(t), \dot{y}_\varepsilon(t)\big) + \partial_t G(\varepsilon, t)\Big)\Big|_{\varepsilon=0} = 0\,, \quad (118)$$

with its associated additional first integral

$$\dot{y}\big(f(x)\dot{y} - y\dot{x}f'(x)\big) + \frac{1}{2}y^2\big(\dot{x}^2 f''(x) - g(x)f'(x) + 2f(x)g'(x)\big)\,. \quad (119)$$

Notice that the gauge term (117) depends on $\dot{x}$.

Among the known functions g that give rise to an isochronous systems one is

$$g(x) = 2\omega^2\Big(1 - \frac{1}{\sqrt{1+x}}\Big)\,, \quad (120)$$

which is a solution of (116) for the choice $f(x) = 1 + x$. Another isochronous case is

$$g(x) = \frac{\omega^2}{4}\Big(1 + x - \frac{1}{(1+x)^3}\Big) \quad (121)$$

which can be obtained for the choice $f(x) = (1+x)^2$. One more isochronous system is given by $g(x) = \frac{d}{dx}(1 - h(x))^2$, where

$$h(x) = \frac{-1 - x + \sqrt{1 - 2x - 3x^2}}{2}\,; \quad (122)$$

this g is a solution of (116) for the choice $f(x) = 3x^2 + 2x - 1$. A fourth is given by $g(x) = \frac{d}{dx}(1 - h(x))^2$, where $h(x) = 1 - \sqrt{1 + 2x - x^2}$; this g is a solution of (116) for the choice $f(x) = x^2 - 2x - 1$.

8 Particle in a plane-wavelike external field

The following time-dependent Lagrangian is taken from a paper [4] by Bobillo-Ares:

$$L(t, q, \dot{q}) = \frac{1}{2}\|\dot{q}\|^2 - V\big(q - tu\big), \qquad q, \dot{q} \in \mathbb{R}^n, \tag{123}$$

where u is a fixed vector in $\mathbb{R}^n$ and V a smooth potential. The associated Lagrange equation is

$$\ddot{q} + \nabla V\big(q - tu\big) = 0 \tag{124}$$

In terms of the energy

$$E(t, q, \dot{q}) = \partial_{\dot{q}} L(t, q, \dot{q}) \cdot \dot{q} - L(t, q, \dot{q}) = \frac{1}{2}\|\dot{q}\|^2 + V\big(q - tu\big)\,, \tag{125}$$

it is easy to check that $\dot{q} \cdot u - E$ is a first integral. Let us see how we can deduce it from Noether's theorem in our framework. Starting from a smooth $q(t)$ and following Bobillo-Ares, we introduce the following space and time changes:

$$q_\varepsilon(t) = q(t) + \varepsilon u, \qquad \tau(\varepsilon, t) := t + \varepsilon\,. \tag{126}$$

Let us try infinitesimal invariance:

$$\begin{aligned}
&\frac{\partial}{\partial \varepsilon}\Big(L\big(\xi, q_\varepsilon(\xi), \dot{q}_\varepsilon(\xi)\big)\Big|_{\xi=\tau(\varepsilon,t)} \partial_t \tau(\varepsilon, t)\Big) = \\
&\quad = \frac{\partial}{\partial \varepsilon}\Big(\frac{1}{2}\|\dot{q}(t+\varepsilon)\|^2 - V\big(q(t+\varepsilon) + \varepsilon u - (t+\varepsilon)u\big)\Big) = \\
&\quad = \dot{q}(t+\varepsilon) \cdot \ddot{q}(t+\varepsilon) - \frac{\partial}{\partial \varepsilon} V\big(q(t+\varepsilon) - tu\big) = \\
&\quad = \dot{q}(t+\varepsilon) \cdot \ddot{q}(t+\varepsilon) - \nabla V\big(q(t+\varepsilon) - tu\big) \cdot \dot{q}(t+\varepsilon) = \\
&\quad = \dot{q}(t+\varepsilon) \cdot \Big(2\ddot{q}(t+\varepsilon) - \Big(\ddot{q}(t+\varepsilon) + \nabla V\big(q(t+\varepsilon) - tu\big)\Big)\Big) = \\
&\quad = \frac{\partial}{\partial t}\|\dot{q}(t+\varepsilon)\|^2 - \dot{q}(t+\varepsilon) \cdot \Big(\ddot{q}(t+\varepsilon) + \nabla V\big(q(t+\varepsilon) - tu\big)\Big)\,.
\end{aligned}$$

When $\varepsilon = 0$ this expression becomes

$$\begin{aligned}
&\frac{\partial}{\partial \varepsilon}\Big(L\big(\xi, q_\varepsilon(\xi), \dot{q}_\varepsilon(\xi)\big)\Big|_{\xi=\tau(\varepsilon,t)} \partial_t \tau(\varepsilon, t)\Big)\Big|_{\varepsilon=0} = \\
&\qquad\qquad = \frac{d}{dt}\|\dot{q}(t)\|^2 - \dot{q}(t) \cdot \Big(\ddot{q}(t) + \nabla V\big(q(t) - tu\big)\Big)\,.
\end{aligned} \tag{127}$$

which further reduces to

$$\frac{\partial}{\partial \varepsilon}\Big(L\big(\xi, q_\varepsilon(\xi), \dot{q}_\varepsilon(\xi)\big)\Big|_{\xi=\tau(\varepsilon,t)} \partial_t \tau(\varepsilon, t)\Big)\Big|_{\varepsilon=0} = \frac{d}{dt}\|\dot{q}(t)\|^2\,. \tag{128}$$

if $q(t)$ solves Lagrange equations (124). This means that we have infinitesimal invariance as in formula (28) with the following choice of gauge function:

$$G(\varepsilon, t) := -\varepsilon\|\dot{q}(t)\|^2\,, \tag{129}$$

and the first integral (48) given by Noether's theorem is

$$\Big(\partial_{\dot q}L\big(t,q(t),\dot q(t)\big)\cdot\partial_\varepsilon q_\varepsilon(t)+L\big(t,q(t),\dot q(t)\big)\partial_\varepsilon\tau(\varepsilon,t)+\partial_\varepsilon G(\varepsilon,t)\Big)\Big|_{\varepsilon=0}= \\ =\dot q(t)\cdot u+L(t,q(t),\dot q(t))-\|\dot q(t)\|^2\,. \quad (130)$$

as expected. We do not know another example where it is natural enough to have infinitesimal invariance with space change, time change and gauge, all all them nontrivial. Also, notice that the gauge G depends on $\dot q$, so that is not of the more familiar form $\varepsilon\cdot\gamma(t,q(t))$.

According to Theorem 1, if we define the additional time change and gauge

$$\mathcal{G}(\varepsilon,t):=\varepsilon\cdot L\big(t,q(t),\dot q(t)\big)\big(\partial_\varepsilon\tau(\varepsilon,t)|_{\varepsilon=0}\big)=\varepsilon\cdot L\big(t,q(t),\dot q(t)\big)\,, \quad (131)$$

$$\mathcal{T}(\varepsilon,t):=\varepsilon\frac{\partial_\varepsilon G(0,t)}{L\big(t,q(t),\dot q(t)\big)}=-\varepsilon\frac{\|\dot q(t)\|^2}{L\big(t,q(t),\dot q(t)\big)}\,, \quad (132)$$

we can attain infinitesimal invariance also with either of the alternative choices

$$\tau_1(\varepsilon,t):=\tau(\varepsilon,t)+\mathcal{T}(\varepsilon,t)\,,\qquad G_1(\varepsilon,t)\equiv 0\,, \quad (133)$$

and

$$\tau_2(\varepsilon,t):=t\,,\qquad G_2(\varepsilon,t):=G(\varepsilon,t)+\mathcal{G}(\varepsilon,t)\,, \quad (134)$$

for the same space change q_ε.

A possible alternative choice for the gauge term in (126) is the following

$$G_3(\varepsilon,t):=\begin{cases}-\big\|q(t+\varepsilon)-q(t)\big\|^2/\varepsilon & \text{if } \varepsilon\neq 0\\ 0 & \text{if } \varepsilon=0\,,\end{cases} \quad (135)$$

which is not linear in ε, and is not a point function of $q(t),\dot q(t)$.

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