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Nonlinear thin-walled beams with a rectangular cross-section - Part I

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Abstract

Our aim is to rigorously derive a hierarchy of one-dimensional models for thin-walled beams with rectangular cross-section, starting from three-dimensional nonlinear elasticity. The different limit models are distinguished by the different scaling of the elastic energy and of the ratio between the sides of the cross-section. In this paper we report the first part of our results. More precisely, denoting by h and δ_h the length of the sides of the cross-section, with $\delta_h \ll h$, and by ε_h^2 the scaling factor of the bulk elastic energy, we analyse the cases in which $\delta_h/\varepsilon_h \to 0$ (subcritical) and $\delta_h/\varepsilon_h \to 1$ (critical).

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Keywords: thin-walled cross-section beams, nonlinear elasticity, Γ-convergence, dimension reduction

1 Introduction

Geometrically, a thin-walled beam is a slender structural element whose length is much larger than the diameter of the cross-section which, on its hand, is larger than the thickness of the thin wall. This kind of beams have been used for a long time in civil and mechanical engineering and, most of all, in flight vehicle structures because of their high ratio between maximum strength and weight.

Because of their slenderness thin-walled beams are quite easy to buckle and to deform and hence, in several circumstances, their study has to be conducted by means of nonlinear theories.

The purpose of this paper is to rigorously derive a hierarchy of one-dimensional models for a thin-walled beam, starting from three-dimensional nonlinear

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elasticity, in the spirit of what has been done in [11] for plates. The different limit models are distinguished by the different scaling of the elastic energy, which, in turn, depends on the scaling of the applied loads.

More precisely, we shall consider a beam of length ℓ with a rectangular cross-section of sides h and δ_h. Let $\Omega_h := (0,\ell) \times (-h/2, h/2) \times (-\delta_h/2, \delta_h/2)$. To model a thin-walled beam we shall require that

$$h \to 0 \quad \text{and} \quad \frac{\delta_h}{h} \xrightarrow{h\to 0} 0.$$

The three-dimensional nonlinear elastic energy (per unit cross-section) associated with a deformation $v : \Omega_h \to \mathbb{R}^3$ is given by

$$E^h(v) = \frac{1}{h\delta_h} \int_{\Omega_h} W(\nabla v(z))\, dz,$$

where W is the elastic energy density of the material. As classical in dimension reduction problems, we rescale the domain Ω_h to a fixed domain $\Omega := (0,\ell) \times (-1/2, 1/2) \times (-1/2, 1/2)$ and, accordingly, we rescale deformations by setting $y(x) := v(x_1, hx_2, \delta_h x_3)$ for every $x \in \Omega$. After this change of variables the elastic energy rewrites as

$$I^h(y) = \int_\Omega W(\nabla_h y(x))\, dx, \quad \text{with} \quad \nabla_h y = \left(y_{,1}\, , \frac{y_{,2}}{h}, \frac{y_{,3}}{\delta_h} \right),$$

where $\nabla_h y$ denotes the rescaled deformation gradient.

We consider a sequence of deformations (y^h) for which the energy scales as ε_h^2, where (ε_h) is a sequence of positive numbers; more precisely, we assume that

$$I^h(y^h) \le C\varepsilon_h^2.$$

This bound is, for instance, satisfied by sequences of global minimizers of the total energies

$$J^h(y) = \int_\Omega W(\nabla_h y)\, dx - \int_\Omega f^h \cdot y\, dx,$$

assuming the applied loads f^h to be of order ε_h^2 (see Section 6). The asymptotic behaviour of the sequence (y^h), as $h \to 0$, is described by the limit of the functionals I^h/ε_h^2 in the sense of Γ-convergence (we refer to [6] for a complete treatment of Γ-convergence). The expression of the Γ-limit depends on the behaviour of ε_h with respect to the intrinsic scale δ_h. More precisely, we can identify three main regimes:

- subcritical: $\dfrac{\delta_h}{\varepsilon_h} \xrightarrow{h\to 0} 0$;
- critical: $\dfrac{\delta_h}{\varepsilon_h} \xrightarrow{h\to 0} 1$;
- supercritical: $\dfrac{\delta_h}{\varepsilon_h} \xrightarrow{h\to 0} +\infty$.

In this paper, in which we report the first part of our studies, we focus on the subcritical and the critical regimes. The supercritical regime will be studied in a forthcoming paper.

In the subcritical regime it is easy to see that, if $\varepsilon_h = O(1)$, then the Γ-limit of the functionals I^h/ε_h^2 exists, up to subsequences, and coincides with the non-linear string model deduced in [1] for a beam of uniformly small cross-section. If instead $\lim_{h\to 0}\varepsilon_h = 0$, we show (Theorem 4.1) that, under appropriate assumptions on W, the Γ-limit of I^h/ε_h^2 is given by

$$I_{sub}(y) = \begin{cases} 0 & \text{if } y \in W^{1,2}((0,\ell),\mathbb{R}^3),\ |y_{,1}| \le 1, \\ +\infty & \text{elsewhere.} \end{cases}$$

The most technical part in the proof of this result is the limsup inequality, where the definition of the recovery sequence is based on the construction of a two dimensional isometry with prescribed Dirichlet and Neumann boundary data on $[0,\ell]$, see Lemma 4.6. The functional I_{sub} denotes the energy of an inextensible string, or more precisely, since the string can grow shorter but not longer (because of the constraint $|y_{,1}| \le 1$), I_{sub} denotes, to use a terminology introduced by Pipkin [17], the energy of an inextensible string with slack.

In the critical regime, again under appropriate conditions on W and under the additional assumption that $\lim_{h\to 0} h^2/\delta_h = 0$, we show (Theorem 5.1) that the Γ-limit of I^h/ε_h^2 is finite only in the class

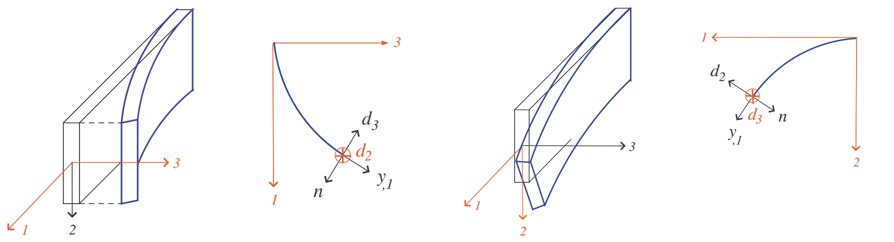
$$\mathcal{A} := \big\{(y,d_2,d_3) \in W^{2,2}((0,\ell);\mathbb{R}^3) \times W^{1,2}((0,\ell);\mathbb{R}^3) \times W^{1,2}((0,\ell);\mathbb{R}^3) : \\ (y_{,1}|d_2|d_3) \in SO(3) \text{ and } y_{,1}\cdot d_{2,1} = 0 \text{ a.e. in } (0,\ell)\big\},$$

and is given by

$$I_{crit}(y,d_2,d_3) = \begin{cases} \dfrac{1}{24}\displaystyle\int_0^\ell Q_2(y_{,1}\cdot d_{3,1}, d_2\cdot d_{3,1})\,dx_1 & \text{if } (y,d_2,d_3) \in \mathcal{A}, \\ +\infty & \text{elsewhere,} \end{cases}$$

where Q_2 is a quadratic form which can be easily computed from the knowledge of W (see (3)). The proof of the liminf inequality deeply relies on the rigidity estimate obtained by Friesecke, James, and Müller [10]. The assumption $\lim_{h\to 0} h^2/\delta_h = \lim_{h\to 0}\varepsilon_h/(\delta_h/h)^2 = 0$ is crucial in the construction of the recovery sequence. Heuristically, it allows us to stretch the mid-plane, i.e., the x_1x_2-plane, by deformations of order $\varepsilon_h/(\delta_h/h)^2$. When $\lim_{h\to 0}\varepsilon_h/(\delta_h/h)^2 \neq 0$ the mid-plane must undergo a deformation which is very close to an isometry. For this reason we conjecture that, in this range, the Γ-limit should coincide with the Γ-limit of the non-linear Kirchhoff functional for a rectangular plate, representing the mid-plane of the beam, when the length of one of the two sides approaches zero. In Remark 5.2 we tried to explain the difficulties we encountered in the case $\lim_{h\to 0}\varepsilon_h/(\delta_h/h)^2 = +\infty$. The functional I_{crit} denotes the energy of a Cosserat thin-walled beam, where $d_2(x_1)$ and $d_3(x_1)$ are the directors which characterize the configuration of the material section of coordinate x_1.

The fact that $(y_{,1}|d_2|d_3) \in SO(3)$ implies that the material section remains at first order orthogonal to the axis of the beam. The warping of the cross-section with respect to the normal plane emerges as a higher order effect (of order $h\delta_h$). The quantities $y_{,1} \cdot d_{3,1}$ and $y_{,1} \cdot d_{2,1}$ may be interpreted (see Antman [2, §8-6]) as the flexural strains around the axis x_2 and x_3, respectively, while the quantity $d_2 \cdot d_{3,1}$ may be interpreted as the torsional strain. The new condition $y_{,1} \cdot d_{2,1} = 0$ implies therefore that there is no flexion around the axis x_3. Since $y_{,1} \cdot d_2 = 0$, the constraint rewrites as $y_{,11} \cdot d_2 = 0$, and, when $y_{,11} \neq 0$, it can be rewritten, by using a Frenet frame, as $\kappa n \cdot d_2 = 0$ where κ is the curvature and n is the normal to the curve y.



Case compatible with $n \cdot d_2 = 0$ This case cannot happen: $n \cdot d_2 = \pm 1$

Γ-convergence results for thin-walled beams were obtained within the theory of linear elasticity in [7, 8, 9], while Γ-convergence results for beams within the nonlinear framework were deduced in [1, 15, 16, 18, 19]. A very different approach to nonlinear rod equations, based on centre manifold theory, was pursued by Mielke [14]. We also mention [4, 5], where a constrained membrane theory analogous to I_{sub} is derived by means of careful approximation results for short maps, that are close in spirit to the analysis performed in Section 4.

The paper is organized as follows. In Section 2 we describe the setting of the problem. In Section 3 we prove some general compactness properties for deformations having equibounded energies. Section 4 is devoted to the subcritical case, while Section 5 to the critical regime. Finally, in Section 6 we introduce applied loads and discuss convergence of minimizers.

Notation. Throughout this article, and unless otherwise stated, we index vector and tensor components as follows: Greek indices α, β and γ take values in the set $\{1,2\}$ and Latin indices i, j, k, l in the set $\{1,2,3\}$. With (e_1, e_2, e_3) we shall denote the canonical basis of $\mathbb{R}^3$. The component k of a vector v will be denoted either by $(v)_k$ or v_k and an analogous notation will be used to denote tensor components. Given two vectors $v, w \in \mathbb{R}^3$, we denote their vector product by $v \times w$. The notation $u_{,k}$ is used to denote the derivative of a scalar or vector function u with respect to the k-th variable. If u is a function of one variable, then the derivative will be denoted by u' or $u_{,1}$ indifferently. $L^p(A;B)$ and $W^{m,p}(A;B)$ are the standard Lebesgue and Sobolev spaces of functions defined on the domain A and taking values in B, with the usual norms $\|\cdot\|_{L^p(A;B)}$ and $\|\cdot\|_{W^{m,p}(A;B)}$, respectively. When $B=\mathbb{R}$ or when the right set B is clear from the context, we will simply write $L^p(A)$ or $W^{m,p}(A)$, sometimes even in the notation used for norms. Convergence in the norm, that is the so-called strong convergence, will be denoted by $\to$ while weak convergence is denoted with $\rightharpoonup$. With a little abuse of language, and because this is a common practice and does not give rise to any confusion, we use to call "sequences" even those families indicized by a continuous parameter h which, throughout the whole paper, will be assumed to belong to the interval $(0,1]$. Finally, all throughout the paper, the constant C in any statement may change line by line. For $A\in\mathbb{R}^{3\times3}$ we denote the Euclidean norm (with the summation convention) by $|A|=\sqrt{tr(AA^T)}=\sqrt{a_{ij}a_{ij}}$. Accordingly, the distance of A from $SO(3)=\{Q\in\mathbb{R}^{3\times3}\ :\ Q^TQ=I,\,\det Q=1\}$ is

$$\operatorname{dist}(A,SO(3))=\inf\{|A-R|\ :\ R\in SO(3)\}.$$

2 Setting of the problem

Let

$$\Omega_h:=(0,\ell)\times\omega_h\subset\mathbb{R}^3,$$

where

$$\omega_h:=\{(z_2,z_3)\ :\ |z_2|<h/2,\,|z_3|<\delta_h/2\}\subset\mathbb{R}^2$$

with $h>0$, $\delta_1:=1$ and

$$\lim_{h\to0}\frac{\delta_h}{h}=0.$$

Henceforth we shall refer to Ω_h as the reference configuration of a three-dimensional body and denote the elastic energy (per unit cross-section) associated with a deformation $v:\Omega_h\to\mathbb{R}^3$ by

$$E^h(v):=\frac{1}{h\delta_h}\int_{\Omega_h}W(\nabla v(z))\,dz.$$

We assume that the stored energy density $W:\mathbb{R}^{3\times3}\to[0,+\infty]$ satisfies the following assumptions:

1. $W\in C^0(\mathbb{R}^{3\times3})$, W is of class C^2 in a neighborhood of $SO(3)$;
2. W is frame indifferent, i.e., $W(F)=W(RF)$ for every $F\in\mathbb{R}^{3\times3}$ and $R\in SO(3)$;
3. $W(F)\geq C\operatorname{dist}^2(F,SO(3))$, $C>0$; $W(F)=0$ if $F\in SO(3)$.

A key role will be played by the following quadratic form:

$$Q_3(F):=\frac{\partial^2W}{\partial F^2}(I)(F,F)=\sum_{i,j,k,l=1}^{3}\frac{\partial^2W}{\partial F_{ij}\partial F_{kl}}(I)F_{ij}F_{kl},\qquad F\in\mathbb{R}^{3\times3}.$$

In view of 3 this form is positive semi-definite and hence convex. Moreover, by 1 and 2 we have that (see, e.g., [12, Section 29])

$$Q_3(F) = Q_3\Big(\frac{F+F^T}{2}\Big). \tag{1}$$

In the special case when the energy density W is isotropic, that is, $W(RFQ) = W(F)$ for all $F \in \mathbb{R}^{3\times 3}$ and $R, Q \in SO(3)$, it turns out that

$$Q_3(F) = 2\mu|e|^2 + \lambda(\operatorname{tr} e)^2, \qquad e = \frac{F+F^T}{2} \tag{2}$$

for some $\lambda, \mu \in \mathbb{R}$.

One of the limit problems will be stated in terms of the function

$$Q_2(\alpha,\beta) := \min\{Q_3(A) \; : \; A \in \mathbb{R}^{3\times 3},\, A^T = A,\, A_{11} = \alpha,\, A_{12} = \beta\}. \tag{3}$$

Let us remark that Q_2 is a positive definite quadratic form. Moreover, in the isotropic case where Q_3 takes the form (2), a simple computation shows that

$$Q_2(\alpha,\beta) = 4\mu\beta^2 + E\alpha^2,$$

where $E := \mu\frac{2\mu+3\lambda}{\mu+\lambda}$ is the Young modulus of the material.

An immediate consequence of assumption 1 is that, by expanding W around the identity, we get

$$W(I+A) = \frac{1}{2}Q_3(A) + \eta(A), \qquad \lim_{|A|\to 0}\frac{\eta(A)}{|A|^2} = 0. \tag{4}$$

To state our results it is convenient to stretch the domain Ω_h along the transverse directions z_2 and z_3 in a way that the transformed domain does not depend on h. Let us therefore set $\omega := \omega_1$, $\Omega := \Omega_1$, and let

$$p_h : \Omega \to \Omega_h$$

be defined by

$$p_h(x) = p_h(x_1, x_2, x_3) = (x_1, hx_2, \delta_h x_3).$$

For every $y \in W^{1,2}(\Omega;\mathbb{R}^3)$ we define the scaled gradient of y as

$$\nabla_h y := \left(y_{,1}\,, \frac{y_{,2}}{h}, \frac{y_{,3}}{\delta_h} \right),$$

where $y_{,i}$ denotes the column vector of the partial derivatives of y with respect to x_i, $i = 1,2,3$. Then we can consider the rescaled energy $I^h : W^{1,2}(\Omega;\mathbb{R}^3) \to [0,+\infty]$ defined by $I^h(y) := E^h(y \circ p_h^{-1})$, i.e.,

$$I^h(y) = \int_\Omega W(\nabla_h y(x))\, dx$$

for every $y \in W^{1,2}(\Omega;\mathbb{R}^3)$.

3 Compactness and lower bound lemmata

In this section we establish some general compactness properties for sequences of deformations with equibounded energy and a lower bound for their energies.

Throughout this section, and the rest of the paper, (ε_h) will denote a sequence of strictly positive real numbers.

A key ingredient in the proof of the compactness properties is the following rigidity result proven by Friesecke, James, and Müller in [10].

Theorem 3.1 *Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Then there exists a constant $C(U)$ with the following property: for every $v \in W^{1,2}(U;\mathbb{R}^n)$ there is an associated rotation $R \in SO(n)$ such that*

$$\|\nabla v - R\|_{L^2} \leq C(U)\|\operatorname{dist}(\nabla v, SO(n))\|_{L^2}.$$

Owing to the previous theorem, we can deduce the following approximation result.

Theorem 3.2 *Let (y^h) be a sequence in $W^{1,2}(\Omega;\mathbb{R}^3)$ such that*

$$\left(\int_\Omega \operatorname{dist}^2(\nabla_h y^h, SO(3))\,dx\right)^{\frac{1}{2}} \leq C\varepsilon_h \tag{5}$$

for every $h > 0$. Then there exist a subsequence (not relabeled) and a corresponding sequence of piecewise constant rotations $R^h : (0,\ell) \times (-1/2, 1/2) \to SO(3)$ such that

$$\|\nabla_h y^h - R^h\|_{L^2} \leq C\varepsilon_h \tag{6}$$

for any h. Moreover, setting $x' = (x_1, x_2)$, there exists a constant C such that

$$\left(\int_{V'} |R^h(x'+\xi) - R^h(x')|^2\,dx'\right)^{\frac{1}{2}} \leq C\big((|\xi_1| \vee h|\xi_2|) + \delta_h\big)\frac{\varepsilon_h}{\delta_h} \tag{7}$$

for any h, any open set V' compactly contained in $V := (0,\ell) \times (-1/2, 1/2)$ and any $\xi = (\xi_1, \xi_2)$ such that $|\xi| < \operatorname{dist}(V', \partial V)$.

Proof. The proof follows the lines of [10, Proof of Theorem 4.1]. The main idea is to consider the unscaled domain Ω_h and to decompose it into a union of small parallelepipeds whose sides are approximately of order δ_h. On each of these subsets one can apply Theorem 3.1 using the same rigidity constant, since one can show that $C(U)$ is invariant under dilations and uniform bi-Lipschitz transformations of U. The main difference with respect to [10] is that, because of the scaling of order h in the x_2 variable, the variations of the approximating rotations in x_1 and in x_2 turn out to have a different order of decay, as $h \to 0$ (see (7)).

Without loss of generality we can assume that $\ell = 1$.

Let us set $v^h := y^h \circ p_h^{-1} : \Omega_h \to \mathbb{R}^3$, that is,

$$v^h(z_1, z_2, z_3) = y^h(z_1, \frac{z_2}{h}, \frac{z_3}{\delta_h}).$$

Hence $\nabla v^h = \nabla_h y^h$. Let us define

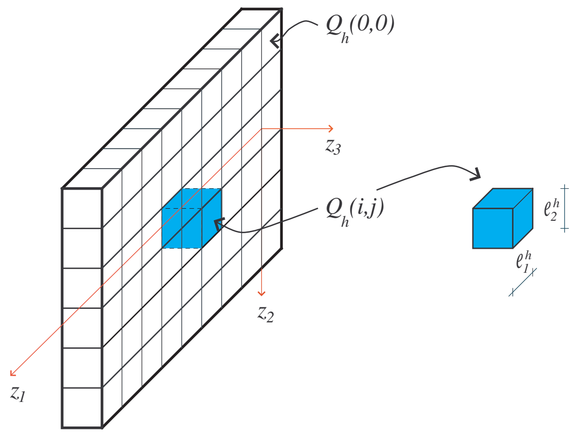
$$N_1^h := [1/\delta_h], \quad N_2^h := [h/\delta_h], \quad \ell_1^h := \frac{1}{N_1^h}, \quad \ell_2^h := \frac{h}{N_2^h},$$

where $[\,\cdot\,]$ stands for the integer part, and consider the family of pairwise disjoint subsets of Ω_h

$$Q_h(i,j) := ((i-1)\ell_1^h, i\ell_1^h) \times (-\frac{h}{2} + (j-1)\ell_2^h, -\frac{h}{2} + j\ell_2^h) \times (-\frac{\delta_h}{2}, \frac{\delta_h}{2})$$

where $i = 1, \dots, N_1^h$ and $j = 1, \dots, N_2^h$. Up to a negligible set, we have that

$$\Omega_h = \bigcup_{i=1}^{N_1^h} \bigcup_{j=1}^{N_2^h} Q_h(i,j).$$



It is now easy to check that each set $Q_h(i,j)$ is the image through a suitable translation and a δ_h-dilation of the set

$$(0, \ell_1^h/\delta_h) \times (0, j\ell_2^h/\delta_h) \times (0, 1).$$

Since $\ell_\alpha^h/\delta_h \to 1$ for $\alpha = 1, 2$, those sets are, in turn, bi-Lipschitz equivalent to the unit cube $(0,1)^3$ with uniformly controlled Lipschitz constant. Therefore, by [11, Theorem 5] the same value of the rigidity constant serves for every set $Q_h(i,j)$. Hence, applying Theorem 3.1 to the functions v^h restricted to the sets $Q_h(i,j)$ we have that there exists a family of rotations $R^h(i,j) \in SO(3)$ such that

$$\int_{Q_h(i,j)} \left|\nabla v^h(z) - R^h(i,j)\right|^2 dz \le C \int_{Q_h(i,j)} \operatorname{dist}^2\left(\nabla v^h(z), SO(3)\right) dz \qquad (8)$$

and the constant C is independent of i, j and h. Then, there exists a piecewise constant map $\hat{R}^h : (0,1)\times(-\frac{h}{2},\frac{h}{2}) \to SO(3)$ such that

$$\begin{aligned}\int_{\Omega_h} |\nabla v^h(z) - \hat{R}^h(z)|^2 dz &\leq \sum_{i,j} \int_{Q_h(i,j)} |\nabla v^h(z) - R^h(i,j)|^2 dz \\ &\leq C \sum_{i,j} \int_{Q_h(i,j)} \operatorname{dist}^2\left(\nabla v^h(z), SO(3)\right) dz \\ &\leq 2C \int_{\Omega_h} \operatorname{dist}^2\left(\nabla v^h(z), SO(3)\right) dz\end{aligned}$$

Passing to the fixed domain Ω, there exists a piecewise constant map $R^h : (0,1)\times(-1/2,1/2) \to SO(3)$ and a constant C such that

$$\int_\Omega |\nabla_h y^h - R^h|^2 dx \leq C \int_\Omega \operatorname{dist}^2\left(\nabla_h y^h, SO(3)\right) dx.$$

This proves the first part of the theorem for the whole sequence y^h.

To prove the second part we have to estimate the variation of $\hat{R}^h$ from a cube to a neighboring one. To this aim, with a slight change of notation we denote by $Q_h(\alpha)$ the cube with side δ_h and center in $(\alpha, 0) \in \mathbb{R}^3$ with $\alpha := (\alpha_1, \alpha_2)$ and $\hat{R}^h(\alpha)$ will be the corresponding rotation. Setting $S_\delta(\alpha)$ the square with side δ and center in α then we have

$$Q_h(\alpha) = S_{\delta_h}(\alpha) \times (-\frac{\delta_h}{2}, \frac{\delta_h}{2}).$$

Let $\beta = (\alpha_1,\alpha_2) + \lambda_1 e_1 + \lambda_2 e_2$ where $\{e_1, e_2\}$ is the canonical basis in $\mathbb{R}^2$ and $\lambda_1, \lambda_2 \in \{0, -\delta_h, \delta_h\}$. Then $S_{\delta_h}(\beta) \subseteq S_{3\delta_h}(\alpha)$. By Theorem 3.1, there exists $\hat{R}^{(3\delta_h)}(\alpha) \in SO(3)$ such that

$$\int_{S_{3\delta_h}(\alpha)\times(-\frac{\delta_h}{2},\frac{\delta_h}{2})} |\hat{R}^{(3\delta_h)}(\alpha) - \nabla v^h(z)|^2 dz \leq C \int_{S_{3\delta_h}(\alpha)\times(-\frac{\delta_h}{2},\frac{\delta_h}{2})} \operatorname{dist}^2(\nabla v^h, SO(3))\, dz \tag{9}$$

So

$$\begin{aligned}|S_{\delta_h}(\beta)||\hat{R}^h(\beta) - \hat{R}^{(3\delta_h)}(\alpha)|^2 &\leq \frac{2}{\delta_h} \int_{S_{\delta_h}(\beta)\times(-\frac{\delta_h}{2},\frac{\delta_h}{2})} |\hat{R}^h(\beta) - \nabla v^h(z)|^2 dz + \\ &+ \frac{2}{\delta_h} \int_{S_{3\delta_h}(\alpha)\times(-\frac{\delta_h}{2},\frac{\delta_h}{2})} |\hat{R}^{(3\delta_h)}(\alpha) - \nabla v^h(z)|^2 dz.\end{aligned}$$

Therefore we have, using (8) and (9) , that

$$|S_{\delta_h}(\beta)||\hat{R}^h(\beta) - \hat{R}^{(3\delta_h)}(\alpha)|^2 \leq \frac{C}{\delta_h} \int_{S_{3\delta_h}(\alpha)\times(-\frac{\delta_h}{2},\frac{\delta_h}{2})} \operatorname{dist}^2(\nabla v^h, SO(3))\, dz. \tag{10}$$

Since $|\hat{R}^h(\alpha)-\hat{R}^h(\beta)|^2 \leq 2\big(|\hat{R}^h(\alpha)-\hat{R}^{(3\delta_h)}(\alpha)|^2+|\hat{R}^h(\beta)-\hat{R}^{(3\delta_h)}(\alpha)|^2\big)$, by (10) and its special case $\alpha=\beta$ (that is $\lambda_1=\lambda_2=0$)

$$|S_{\delta_h}(\beta)||\hat{R}^h(\alpha)-\hat{R}^h(\beta)|^2 \leq \frac{C}{\delta_h}\int_{S_{3\delta_h}(\alpha)\times(-\frac{\delta_h}{2},\frac{\delta_h}{2})} \operatorname{dist}^2(\nabla v^h, SO(3))\,dz \tag{11}$$

which also can be written, being $\hat{R}^h$ piecewise constant,

$$\int_{S_{\delta_h}(\alpha)} |\hat{R}^h(z'+\lambda_1 e_1+\lambda_2 e_2)-\hat{R}^h(z')|^2 dz' \leq \frac{C}{\delta_h}\int_{S_{3\delta_h}(\alpha)\times(-\frac{\delta_h}{2},\frac{\delta_h}{2})} \operatorname{dist}^2(\nabla v^h, SO(3))\,dz.$$

Hence for $\eta\in\mathbb{R}^2$ satisfying $|\eta|_\infty := \max\{|\eta\cdot e_1|,|\eta\cdot e_2|\}\leq\delta_h$,

$$\int_{S_{\delta_h}(\alpha)} |\hat{R}^h(z'+\eta)-\hat{R}^h(z')|^2 dz' \leq \frac{C}{\delta_h}\int_{S_{3\delta_h}(\alpha)\times(-\frac{\delta_h}{2},\frac{\delta_h}{2})} \operatorname{dist}^2(\nabla v^h, SO(3))\,dz.$$

Let now V' be an open set compactly contained in $V=(0,1)\times(-1/2,1/2)$ and denote by V'_h and $V_h=(0,1)\times(-\frac{h}{2},\frac{h}{2})$ the corresponding scaled domains. Let us consider a more general translation vector $\eta\in\mathbb{R}^2$ such that $|\eta|_\infty<\operatorname{dist}(V'_h,\partial V_h)$. As now it can happen that $|\eta|_\infty>\delta_h$, then we set $N:=\max\{[\frac{\eta}{\delta_h}\cdot e_1],[\frac{\eta}{\delta_h}\cdot e_2]\}$ and pick $\eta_0,\ldots,\eta_{N+1}$ such that $\eta_0=0$, $\eta_{N+1}=\eta$, $|\eta_{k+1}-\eta_k|_\infty\leq\delta_h$. Then

$$|\hat{R}^h(z'+\eta)-\hat{R}^h(z')|^2 \leq (N+1)\sum_{k=0}^{N}|\hat{R}^h(z'+\eta_{k+1})-\hat{R}^h(z'+\eta_k)|^2$$

and hence

$$\int_{S_{\delta_h}(\alpha)} |\hat{R}^h(z'+\eta)-\hat{R}^h(z')|^2 dz' \leq \frac{C(N+1)}{\delta_h}\sum_{k=0}^{N}\int_{S_{3\delta_h}(\alpha+\eta_k)\times(-\frac{\delta_h}{2},\frac{\delta_h}{2})} \operatorname{dist}^2(\nabla v^h, SO(3))\,dz.$$

Summing over all $S_{\delta_h}(\alpha)\cap V'_h\neq\varnothing$ and using that each $z\in V_h\times(-\frac{\delta_h}{2},\frac{\delta_h}{2})$ is contained in at most $C(N+1)$ of the sets $S_{3\delta_h}(\alpha+\eta_k)\times(-\frac{\delta_h}{2},\frac{\delta_h}{2})$ with C independent of h, then

$$\int_{V'_h} |\hat{R}^h(z'+\eta)-\hat{R}^h(z')|^2 dz' \leq \frac{C}{\delta_h}\Big(\frac{|\eta_1|}{\delta_h}\vee\frac{|\eta_2|}{\delta_h}+1\Big)^2\int_{V_h\times(-\frac{\delta_h}{2},\frac{\delta_h}{2})} \operatorname{dist}^2(\nabla v^h, SO(3))\,dz.$$

The claim follows now by passing to the fixed domain with the usual change of variables, setting $\xi_1=\eta_1$ and $\xi_2=\eta_2/h$ and observing that $|\eta|_\infty<\operatorname{dist}(V'_h,\partial V_h)$ if and only if $|\xi|_\infty<\operatorname{dist}(V',\partial V)$. □

Lemma 3.3 *Under assumption* (5)*, there is a sequence of maps*

$$\tilde{R}^h\in C^\infty([0,\ell]\times[-\tfrac{1}{2},\tfrac{1}{2}];\mathbb{R}^{3\times3})$$

such that, for every $h>0$,

1. $\|\tilde{R}^h - R^h\|_{L^\infty} \leq Ch^{1/2}\varepsilon_h/\delta_h$*,*

2. $\|\nabla_h y^h - \tilde{R}^h\|_{L^2} \leq C\varepsilon_h$*,*

3. $\|\tilde{R}^h_{,1}\|_{L^2} \leq C\varepsilon_h/\delta_h$*,* $\quad \|\tilde{R}^h_{,2}\|_{L^2} \leq Ch\varepsilon_h/\delta_h$*,*

where the constant C may change from line to line. Moreover, if in addition $h^{1/2}\varepsilon_h/\delta_h \to 0$, then we can take

4. $\tilde{R}^h(x_1, x_2) \in \mathrm{SO}(3)$ *for every* $(x_1, x_2) \in (0, \ell) \times (-1/2, 1/2)$ *and every* $h > 0$*.*

Proof. The strategy of the proof is the same as in [11, Theorem 6]: we consider the approximating rotations constructed in Theorem 3.2 and regularize them on a small scale. The difference quotient estimate (7) translates then into suitable bounds for the derivatives of the mollifications. In contrast with what is done in [11], we need to choose mollifiers that have a different scaling in the x_1 and x_2 variables, because of the different order of decay of the variations of R^h in the two variables.

Let (R^h) denote the sequence of rotations of the previous Theorem 3.2 extended by successive reflections to the whole of $\mathbb{R}^2$. Let η_h be any sequence of mollifiers which will be made precise in the following and

$$\bar{R}^h(y') := R^h * \eta_h(y') = \int \eta_h(z') R^h(y' - z')\, dz'.$$

Let $V' \subset\subset V = (0, \ell) \times (-1/2, 1/2)$. Using the fact that $\int \eta_h = 1$ and Hölder's inequality, we observe that

$$\begin{aligned} \|\bar{R}^h - R^h\|^2_{L^2(V')} &= \int_{V'} \Big| \int \eta_h(z') \big(R^h(y' - z') - R^h(y')\big)\, dz' \Big|^2 dy' \\ &\leq \int |\eta_h(z')|^2 dz' \int_{\operatorname{supp} \eta_h} \int_{V'} |R^h(y' - z') - R^h(y')|^2\, dy'\, dz'. \end{aligned} \tag{12}$$

Let us now choose the sequence of mollifiers as follows: for $i = 1, 2$, let $\eta_i \in C^\infty_c(-1/2, 1/2)$, $\eta_i \geq 0$, $\int \eta_i = 1$ and define

$$\eta(y') := \eta_1(y_1)\eta_2(y_2), \qquad \eta_h(z') := \frac{h}{\delta_h^2} \eta\Big(\frac{z_1}{\delta_h}, \frac{hz_2}{\delta_h}\Big) = \frac{h}{\delta_h^2} \eta_1\Big(\frac{z_1}{\delta_h}\Big)\eta_2\Big(\frac{hz_2}{\delta_h}\Big).$$

Then $\eta_h \in C^\infty_c((-\delta_h/2, \delta_h/2) \times (-\delta_h/2h, \delta_h/2h))$ and $\int \eta_h = 1$. In particular, for any h small enough, $\operatorname{supp} \eta_h$ is contained into a ball with radius smaller than the distance from V' to ∂V; therefore we can apply estimate (7) in (12) and substitute the expression of η_h, so obtaining

$$\begin{aligned} \|\bar{R}^h - R^h\|^2_{L^2(V')} &\leq C \int |\eta_h(z')|^2 dz' \int_{\operatorname{supp} \eta_h} \big((|z_1| \vee h|z_2|) + \delta_h\big)^2 \varepsilon_h^2 \delta_h^{-2}\, dz' \\ &\leq C \int \Big|\frac{h}{\delta_h^2} \eta(x')\Big|^2 \frac{\delta_h^2}{h} dx' \int_{\operatorname{supp} \eta_h} \delta_h^2 \varepsilon_h^2 \delta_h^{-2}\, dz' \leq C \frac{h}{\delta_h^2} \varepsilon_h^2 \frac{\delta_h^2}{h} = C\varepsilon_h^2 \end{aligned}$$

which implies that

$$\|\bar{R}^h - R^h\|_{L^2(V)} \leq C\varepsilon_h \tag{13}$$

since the constant C does not depend on the choice of V'.

Applying Hölder's inequality and proceeding as above, we have

$$\begin{aligned}
\int_{V'} |\bar{R}^h_{,1}|^2 dy' &= \int_{V'} \Big| \int \eta_{h,1}\big(R^h(y'-z') - R^h(y')\big)\, dz' \Big|^2 dy' \\
&\leq \int_{\operatorname{supp}\eta_h} |\eta_{h,1}|^2 dz' \int_{V'} \int_{\operatorname{supp}\eta_h} |R^h(y'-z') - R^h(y')|^2\, dz' dy' \\
&\leq C\frac{h}{\delta_h^4} \int_{\operatorname{supp}\eta_h} \big((|z_1| \vee h|z_2|) + \delta_h\big)^2 \varepsilon_h^2 \delta_h^{-2}\, dz' \leq C\frac{h}{\delta_h^4}\varepsilon_h^2 \frac{\delta_h^2}{h}
\end{aligned}$$

which implies

$$\|\bar{R}^h_{,1}\|_{L^2} \leq C\frac{\varepsilon_h}{\delta_h}. \tag{14}$$

Analogously we can prove that

$$\|\bar{R}^h_{,2}\|_{L^2} \leq Ch\frac{\varepsilon_h}{\delta_h}. \tag{15}$$

Rewriting inequality (11) with $\hat{R}^h(\eta_1, \eta_2) = R^h(\eta_1, \eta_2/h)$, performing the corresponding change of variables in the integral on the right-hand side, estimating from above with the integral on Ω, and using assumption (5), we have

$$\delta_h^2 |R^h(x_1+\xi_1, x_2+\xi_2) - R^h(x_1,x_2)|^2 \leq Ch \int_\Omega \operatorname{dist}^2(\nabla_h y^h, SO(3))\, dx \leq Ch\varepsilon_h^2$$

for every $|\xi_1| \leq \delta_h$ and $|\xi_2| \leq \delta_h/h$. Then we have

$$|\bar{R}^h(x_1,x_2) - R^h(x_1,x_2)| \leq \int_{\operatorname{supp}\eta_h} |\eta_h(z')||R^h(x'-z') - R^h(x')|\, dz' \leq Ch^{1/2}\varepsilon_h/\delta_h$$

which implies

$$\|\bar{R}^h - R^h\|_{L^\infty} \leq Ch^{1/2}\varepsilon_h/\delta_h. \tag{16}$$

Conditions *1*, *2* and *3* are proven by setting $\tilde{R}^h = \bar{R}^h$. In particular, *2* follows from (6) and (13).

Let now assume that $h^{1/2}\varepsilon_h/\delta_h \to 0$. Let $\Pi : U \to \mathrm{SO}(3)$ be a smooth projection from a compact neighborhood U of SO(3) onto SO(3). Since by (16) the functions $\bar{R}^h$ take values in U for h small enough, we can define $\tilde{R}^h := \Pi\bar{R}^h$. By the regularity of the projection, $\tilde{R}^h$ is smooth. Condition *4* of the statement is satisfied. Claim *1* follows from the continuity of the projection and inequality (16). From (6), (13) and still by continuity of Π we obtain *2*. Finally, claim *3* follows from (14), (15) and the boundedness of the gradient of Π. □

Let us conclude this section by proving a useful lower bound of the energy.

Lemma 3.4 *Assume that* $\lim_{h\to 0}\varepsilon_h = 0$. *Let* $(y^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$, $(R^h) \subset SO(3)$ *and*

$$G^h := \frac{R^h\nabla_h y^h - I}{\varepsilon_h} \rightharpoonup G \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}).$$

Then

$$\liminf_{h\to 0} \frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y^h)\,dx \geq \frac{1}{2}\int_\Omega Q_3(G)\,dx.$$

Proof. The proof is an adaptation of an argument by Friesecke, James, and Müller [10]. By expanding W around the identity as in (4) and setting $\omega(t) = \sup_{|A|\leq t}\{|\eta(A)|/|A|^2\}$, we have

$$W(I+A) \geq \frac{1}{2}Q_3(A) - \omega(|A|)|A|^2,$$

where $\omega(t)\to 0$, as $t\to 0^+$.

By the frame indifference of W then we have

$$\begin{aligned} W(\nabla_h y^h) &= W(R^h\nabla_h y^h) = W(I+\varepsilon_h G^h) \\ &\geq \frac{\varepsilon_h^2}{2}Q_3(G^h) - \varepsilon_h^2\omega(\varepsilon_h|G^h|)|G^h|^2. \end{aligned} \tag{17}$$

From the boundedness of G^h in L^2 it follows that the functions

$$\chi_h(x) := \begin{cases} 1 & \text{if } |G^h(x)| \leq 1/\sqrt{\varepsilon_h}, \\ 0 & \text{otherwise,} \end{cases}$$

converge in measure to the constant 1 and are uniformly bounded. Hence

$$\chi_h G^h \rightharpoonup G \text{ in } L^2. \tag{18}$$

By inequality (17) we obtain

$$\begin{aligned} \frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y^h)\,dx &\geq \frac{1}{\varepsilon_h^2}\int_\Omega \chi_h W(\nabla_h y^h)\,dx \\ &\geq \frac{1}{2}\int_\Omega \chi_h Q_3(G^h)\,dx - \int_\Omega \chi_h\omega(\varepsilon_h|G^h|)|G^h|^2\,dx \\ &= \frac{1}{2}\int_\Omega Q_3(\chi_h G^h)\,dx - \int_\Omega \chi_h\omega(\varepsilon_h|G^h|)|G^h|^2\,dx. \end{aligned}$$

Since Q_3 is a semi-positive definite quadratic form (by the hypotheses on W), hence convex, the first integral on the right-hand side is lower semicontinuous with respect to the convergence (18). The second integral converges to zero, because $|G^h|^2$ is bounded in $L^1(\Omega)$ and $\chi_h\omega(\varepsilon_h|G^h|) \leq \omega(\sqrt{\varepsilon_h})$, which converges to zero. Therefore the claim follows by taking the liminf as $h\to 0$. □

4 The subcritical case

In this section we focus on the study of the asymptotic behaviour of a sequence of deformations $(y^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ satisfying

$$I^h(y^h) = \int_\Omega W(\nabla_h y^h)\,dx \le C\varepsilon_h^2, \tag{19}$$

where

$$\lim_{h\to 0} \varepsilon_h = \lim_{h\to 0} \frac{\delta_h}{\varepsilon_h} = 0. \tag{20}$$

The main result of the section is the following.

Theorem 4.1 *Assume* (20)*. Then the following assertions hold.*

1. **(Compactness)** *If $(y^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ satisfies* (19)*, then there exist some constants $c^h \in \mathbb{R}^3$ and a function $y \in W^{1,2}(\Omega;\mathbb{R}^3)$ such that $|y_{,1}| \le 1$, $y_{,2} = y_{,3} = 0$ a.e. in Ω and $y^h - c^h \rightharpoonup y$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ (up to subsequences).*

2. **(Upper bound)** *For every $y \in W^{1,2}(\Omega;\mathbb{R}^3)$ such that $|y_{,1}| \le 1$, $y_{,2} = y_{,3} = 0$ a.e. in Ω there exists a sequence $(y^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that $y^h \rightharpoonup y$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ and*
$$\lim_{h\to 0} \frac{1}{\varepsilon_h^2} I^h(y^h) = 0.$$

Remark 4.2 Theorem 4.1 implies that, under the assumption (20), the sequence $\left(\frac{1}{\varepsilon_h^2} I^h\right)$ Γ-converges in the weak topology of $W^{1,2}(\Omega;\mathbb{R}^3)$ to the functional

$$I(y) = \begin{cases} 0 & \text{if } |y_{,1}| \le 1,\ y_{,2} = y_{,3} = 0 \text{ a.e. in } \Omega, \\ +\infty & \text{else.} \end{cases}$$

We begin with some approximation lemmata, which will be useful in the proof of the upper bound.

Lemma 4.3 *Let $v \in L^2((0,\ell);\mathbb{R}^3)$ be such that $|v| \le 1$ a.e. in $(0,\ell)$. Then there exists a sequence (v_k) in $C^\infty([0,\ell];\mathbb{R}^3)$ such that $|v_k| = 1$, $v_k' \ne 0$ in $[0,\ell]$, and $v_k \rightharpoonup v$ weakly in $L^2((0,\ell);\mathbb{R}^3)$.*

Proof. We split the proof into three steps.

Step 1. Assume first that v is constant, that is, $v(t) = y_0$ for a.e. $t \in (0,L)$, with $y_0 \in \mathbb{R}^3$, $|y_0| \le 1$. Then there exist $y_1, y_2 \in \mathbb{R}^3$ with $|y_\alpha| = 1$ and $\lambda \in (0,1]$ such that

$$y_0 = \lambda y_1 + (1-\lambda) y_2.$$

As $|y_1| = |y_2| = 1$, we can find two pairs (w_1, z_1), (w_2, z_2) of orthonormal vectors, which are orthogonal to y_1 and y_2, respectively.

Let y, w, $z\colon \mathbb{R} \to \mathbb{R}^3$ be the periodic functions of period ℓ defined on $(0, \ell]$ by

$$\begin{aligned} y(t) &:= \chi_{(0,\lambda\ell]} y_1 + \chi_{(\lambda\ell,\ell]} y_2, \\ w(t) &:= \chi_{(0,\lambda\ell]} w_1 + \chi_{(\lambda\ell,\ell]} w_2, \\ z(t) &:= \chi_{(0,\lambda\ell]} z_1 + \chi_{(\lambda\ell,\ell]} z_2, \end{aligned}$$

where χ denotes the characteristic function. For every $k \in \mathbb{N}$ we consider the function $\tilde{v}_k\colon [0, \ell] \to \mathbb{R}^3$ defined as follows:

$$\tilde{v}_k(t) := \sqrt{1 - a_k^2}\, y(kt) + a_k\big(\cos t\, w(kt) + \sin t\, z(kt)\big),$$

where a_k is a sequence of positive numbers converging to 0.

It is well known that the sequence $y(k\,\cdot)$ converges to v weakly in $L^2((0, \ell); \mathbb{R}^3)$, as $k \to \infty$. Since $a_k \to 0$ and

$$|\cos t\, w(kt) + \sin t\, z(kt)| = 1 \quad \text{for every } t,$$

we conclude that $\tilde{v}_k$ converges to v weakly in $L^2((0, \ell); \mathbb{R}^3)$. Moreover, $|\tilde{v}_k(t)| = 1$ for any $t \in [0, \ell]$. Away from the points

$$t_k^{m,1} := \frac{1}{k}(\lambda + m)\ell, \quad t_k^{m,2} := \frac{1}{k} m\ell, \quad k \in \mathbb{N},\ m = 0, \ldots, k,$$

the function $\tilde{v}_k$ is differentiable and

$$\tilde{v}_k'(t) = a_k\big(- \sin t\, w(kt) + \cos t\, z(kt)\big),$$

hence $|\tilde{v}_k'| = a_k \neq 0$. To conclude it remains to modify $\tilde{v}_k$ around the jump points $t_k^{m,\alpha}$ in such a way to obtain a smooth function. Let $\eta_k > 0$ be such that $k\eta_k \to 0$. For every $k \in \mathbb{N}$, $m = 0, \ldots, k-1$, and $\alpha = 1, 2$, we can construct a function $\phi_k^{m,\alpha} \in C^\infty([-\eta_k, \eta_k]; \mathbb{R}^3)$ such that $|\phi_k^{m,\alpha}| = 1$, $(\phi_k^{m,\alpha})' \neq 0$, and for every $j \in \mathbb{N} \cup \{0\}$

$$D^j \phi_k^{m,\alpha}(-\eta_k) = D^j \tilde{v}_k(t_k^{m,\alpha} - \eta_k), \quad D^j \phi_k^{m,\alpha}(\eta_k) = D^j \tilde{v}_k(t_k^{m,\alpha} + \eta_k),$$

where D^j denotes the derivative of order j. Indeed, as $|\tilde{v}_k(t)| = 1$, there exist two real functions φ_k and ϑ_k such that

$$\tilde{v}_k(t) = \begin{pmatrix} \sin \varphi_k(t) \cos \vartheta_k(t) \\ \sin \varphi_k(t) \sin \vartheta_k(t) \\ \cos \varphi_k(t) \end{pmatrix},$$

where φ_k and ϑ_k are smooth except at the points $t_k^{m,\alpha}$. For every k, m, α and for $t \in [-\eta_k, \eta_k]$ we define

$$\phi_k^{m,\alpha}(t) := \begin{pmatrix} \sin \varphi_k^{m,\alpha}(t) \cos \vartheta_k^{m,\alpha}(t) \\ \sin \varphi_k^{m,\alpha}(t) \sin \vartheta_k^{m,\alpha}(t) \\ \cos \varphi_k^{m,\alpha}(t) \end{pmatrix},$$

where $\varphi_k^{m,\alpha}$ and $\vartheta_k^{m,\alpha}$ are smooth transitions constructed in such a way that $(\phi_k^{m,\alpha})' \neq 0$ and

$$\begin{aligned} D^j \varphi_k^{m,\alpha}(-\eta_k) = D^j \varphi_k(t_k^{m,\alpha} - \eta_k), \quad D^j \varphi_k^{m,\alpha}(\eta_k) = D^j \varphi_k(t_k^{m,\alpha} + \eta_k), \\ D^j \vartheta_k^{m,\alpha}(-\eta_k) = D^j \vartheta_k(t_k^{m,\alpha} - \eta_k), \quad D^j \vartheta_k^{m,\alpha}(\eta_k) = D^j \vartheta_k(t_k^{m,\alpha} + \eta_k) \end{aligned}$$

for every $j \in \mathbb{N} \cup \{0\}$.

The required sequence is then given by

$$v_k(t) := \begin{cases} \tilde{v}_k(t) & \text{if } t \in [0, \ell] \setminus \cup_{m=0}^{k-1} \cup_{\alpha=1}^{2} (t_k^{m,\alpha} - \eta_k, t_k^{m,\alpha} + \eta_k), \\ \phi_k^{m,\alpha}(t - t_k^{m,\alpha}) & \text{if } t \in (t_k^{m,\alpha} - \eta_k, t_k^{m,\alpha} + \eta_k). \end{cases}$$

It is clear that v_k is smooth, $|v_k| = 1$, and $v_k' \neq 0$. Moreover, since $k\eta_k \to 0$, we have that $\|v_k - \tilde{v}_k\|_{L^2} \to 0$, hence v_k converge to v weakly in $L^2((0,\ell);\mathbb{R}^3)$.

Step 2. Let v be a piecewise constant function with $|v| \leq 1$ a.e. in $(0,\ell)$. Then there exist a partition $0 = s_0 < s_1 < \cdots < s_M = \ell$ of $(0,\ell)$ and some constant vectors $\bar{y}_1, \ldots, \bar{y}_M$ with $|\bar{y}_i| \leq 1$ such that

$$v = \sum_{i=1}^{M} \chi_{(s_{i-1},s_i)} \bar{y}_i.$$

On every interval (s_{i-1}, s_i) we can repeat the construction of Step 1. In this way we obtain a sequence (v_k) with the desired properties, except for the regularity, as v_k may jump at the points s_i. Around these points we can make v_k smooth by introducing a transition function like $\phi_k^{m,\alpha}$ of Step 1 on small intervals $(s_{i-1} - \eta_k, s_i + \eta_k)$, where $\eta_k \to 0$.

Step 3. Let us consider now the general case. Let v be as in the statement of the lemma. Then we can construct a sequence $\hat{v}_n$ of piecewise constant functions with $|\hat{v}_n| \leq 1$ a.e. and $\hat{v}_n \to v$ strongly in $L^2((0,\ell);\mathbb{R}^3)$.

We note that, owing to the bound $|v| \leq 1$, we can restrict our attention to a bounded subset of $L^2((0,\ell);\mathbb{R}^3)$, where the weak topology is metrizable. By the previous step for every n there exists a sequence $(v_k^n) \subset C^\infty((0,\ell);\mathbb{R}^3)$ such that $|v_k^n| = 1$, $(v_k^n)' \neq 0$, and $v_k^n \rightharpoonup v_n$ weakly in $L^2((0,\ell);\mathbb{R}^3)$. A diagonal argument allows us to conclude the proof. □

Lemma 4.4 *Let $R \in W^{1,2}((0,\ell);\mathbb{R}^{3\times3})$ be such that $R \in SO(3)$ a.e. in $(0,\ell)$. Then there exists a sequence of analytic functions $R^\nu : [0,\ell] \to \mathbb{R}^{3\times3}$ such that $R^\nu \in SO(3)$ in $[0,\ell]$ and $R^\nu \to R$ in $W^{1,2}((0,\ell);\mathbb{R}^{3\times3})$. If moreover $(R^T R_{,1})_{ij} = 0$ for some $i \neq j$, then we can choose also $((R^\nu)^T R^\nu_{,1})_{ij} = 0$ in $[0,\ell]$ for every $\nu \in \mathbb{N}$. If in addition $R \in C^k([0,\ell];\mathbb{R}^{3\times3})$ for some $k \geq 1$, then $R^\nu \to R$ in $C^k([0,\ell];\mathbb{R}^{3\times3})$.*

Proof. Let us set $A := R^T R_{,1} \in L^2((0,\ell);\mathbb{R}^{3\times3})$. Since $(R^T R)_{,1} = 0$, the matrix A is skew-symmetric. Then there are α, β, $\gamma \in L^2(0,\ell)$ such that

$$A = \begin{pmatrix} 0 & \gamma & \alpha \\ -\gamma & 0 & \beta \\ -\alpha & -\beta & 0 \end{pmatrix}.$$

Let (α^ν), (β^ν), and (γ^ν) be sequences of polynomials on $[0,\ell]$ such that

$$\alpha^\nu \to \alpha, \quad \beta^\nu \to \beta, \quad \gamma^\nu \to \gamma \quad \text{in } L^2(0,\ell) \tag{21}$$

and define

$$A^\nu := \begin{pmatrix} 0 & \gamma^\nu & \alpha^\nu \\ -\gamma^\nu & 0 & \beta^\nu \\ -\alpha^\nu & -\beta^\nu & 0 \end{pmatrix}.$$

Let R^ν be the unique solution to the linear initial value problem

$$\begin{cases} R^\nu_{,1} = R^\nu A^\nu & \text{in } [0,\ell], \\ R^\nu(0) = R(0). \end{cases} \tag{22}$$

Since A^ν is analytic on $[0,\ell]$, the same is true for R^ν. Moreover, from the fact that $({R^\nu}^T R^\nu)_{,1} = {A^\nu}^T + A^\nu = 0$ and $R^\nu(0)^T R^\nu(0) = R(0)^T R(0) = I$, it follows that ${R^\nu}^T R^\nu \equiv I$, that is, $R^\nu(t) \in SO(3)$ for any $t \in [0,\ell]$.

Setting $Z^\nu := R - R^\nu$, it is easy to check that Z^ν is the unique solution to the problem

$$\begin{cases} Z^\nu_{,1} = Z^\nu A^\nu + R(A - A^\nu) & \text{in } [0,\ell], \\ Z^\nu(0) = 0. \end{cases} \tag{23}$$

By Gronwall's lemma, Hölder's inequality and the boundedness of (A^ν) in L^2, for a.e. $t \in (0,\ell)$ we have

$$\begin{aligned} |Z^\nu(t)| &\leq \int_0^t |R(s)||A(s) - A^\nu(s)|\, ds\ e^{\int_0^t |A^\nu(s)|\, ds} \\ &\leq C\|R\|_{L^\infty}\|A - A^\nu\|_{L^2}, \end{aligned}$$

hence

$$|R(t) - R^\nu(t)| \leq C\|A - A^\nu\|_{L^2},$$

which implies that

$$R^\nu \to R \text{ in } L^\infty((0,\ell);\mathbb{R}^{3\times3}). \tag{24}$$

From (23) we have

$$\|Z^\nu_{,1}\|_{L^2} \leq \|Z^\nu\|_{L^\infty}\|A^\nu\|_{L^2} + C\|R\|_{L^\infty}\|A - A^\nu\|_{L^2} \tag{25}$$

and taking the limit as $\nu \to \infty$ we get $\|Z^\nu_{,1}\|_{L^2} \to 0$, which implies that the convergence in (24) is in the norm of $W^{1,2}((0,\ell);\mathbb{R}^{3\times3})$, and the first part of the lemma is proven.

To prove the second part of the statement, let assume, to fix ideas, that $(R^T R_{,1})_{12} = 0$. Then we have $\gamma = 0$, hence we can choose $\gamma^\nu = 0$ for any $\nu \in \mathbb{N}$. Thus, by (22), we have

$$({R^\nu}^T R^\nu_{,1})_{12} = ({R^\nu}^T R^\nu A^\nu)_{12} = A^\nu_{12} = \gamma^\nu = 0.$$

If now $R \in C^k([0,\ell];\mathbb{R}^{3\times3})$, then $A \in C^{k-1}([0,\ell];\mathbb{R}^{3\times3})$. Thus, we can find sequences of polynomials (α^ν), (β^ν), and (γ^ν) such that the convergences in

(21) hold with respect to the C^{k-1} norm. Arguing similarly as in (25), it is then easy to prove that $R^\nu \to R$ in $C^k([0,\ell];\mathbb{R}^{3\times 3})$. $\square$

Remark 4.5 The purpose of the present remark is to give a geometrical-mechanical interpretation of the components of the matrix $A = R^T R_{,1}$, which played a major role in the proof of Lemma 4.4.

Let $R \in W^{1,2}((0,\ell);\mathbb{R}^{3\times 3})$ be such that $R \in SO(3)$ a.e. in $(0,\ell)$, and assume that the vectors $R(x_1)e_2$ and $R(x_1)e_3$ characterize the configuration of the deformed cross-section with coordinate x_1, like, for instance, the vectors e_2 and e_3 characterize the cross-section in the reference configuration. Then, from the identity

$$A_{ij} = e_i \cdot R^T R_{,1} e_j = Re_i \cdot (Re_j)_{,1}$$

we can see that the component A_{ij} measures the change of the characterizing vector Re_j along the direction Re_i. Thus A_{ij} are strains, in particular A_{12} and A_{13} measure fleaxure, while A_{23} measures torsion.

Lemma 4.6 *Let $y, d_2 : [0,\ell] \to \mathbb{R}^3$ be analytic functions such that*

$$|y'| = |d_2| = 1, \quad y'' \neq 0, \quad \textit{and} \quad y' \cdot d_2 = y'' \cdot d_2 = 0 \quad \textit{in } [0,\ell].$$

Then there exist some $\eta > 0$ and an analytic function $u : [0,\ell] \times [-\frac{\eta}{2}, \frac{\eta}{2}] \to \mathbb{R}^3$ such that u is an isometry on $[0,\ell] \times [-\frac{\eta}{2}, \frac{\eta}{2}]$ satisfying $u(x_1, 0) = y(x_1)$ and $u_{,2}(x_1, 0) = d_2(x_1)$ for every $x_1 \in [0,\ell]$.

Proof. We need to show that there exists some tubular neighbourhood U of $[0,\ell]\times\{0\}$ such that the problem

$$\begin{cases} u_{,1} \cdot u_{,1} = 1 & \text{in } U, \\ u_{,2} \cdot u_{,2} = 1 & \text{in } U, \\ u_{,1} \cdot u_{,2} = 0 & \text{in } U, \\ u(x_1, 0) = y(x_1) & \text{for } x_1 \in [0,\ell], \\ u_{,2}(x_1, 0) = d_2(x_1) & \text{for } x_1 \in [0,\ell], \end{cases} \tag{26}$$

has an analytic solution u defined in U.

We follow the classical results in the construction of local isometric embeddings of analytic metrics (see [13, Section 1.1] or [20, Chapter 11]). We first derive an equivalent formulation of (26). Differentiating the second equation in (26) with respect to x_1 we deduce that

$$u_{,12} \cdot u_{,2} = 0 \quad \text{in } U. \tag{27}$$

Differentiating the system with respect to x_2 and using the equality above, we obtain that any solution to (26) satisfies

$$\begin{cases} u_{,1} \cdot u_{,22} = 0 & \text{in } U, \\ u_{,2} \cdot u_{,22} = 0 & \text{in } U, \\ u_{,11} \cdot u_{,22} = u_{,12} \cdot u_{,12} & \text{in } U. \end{cases} \tag{28}$$

We shall show that the system (28), supplemented by the initial conditions

$$u(x_1, 0) = y(x_1), \quad u_{,2}(x_1, 0) = d_2(x_1) \quad \text{for } x_1 \in [0, \ell], \tag{29}$$

is equivalent to (26). Indeed, let u be an analytic solution to (28)–(29). From the second equation in (28) and the assumption that $|d_2| = 1$ it follows immediately that $|u_{,2}| = 1$ in U and, in particular, that (27) is satisfied. Now, using the first equation in (28) and (27), we have

$$\big(u_{,1} \cdot u_{,2}\big)_{,2} = u_{,12} \cdot u_{,2} + u_{,1} \cdot u_{,22} = 0 \quad \text{in } U.$$

From the assumption $y' \cdot d_2 = 0$ on $[0, \ell]$ it follows that also the third equation in (26) is satisfied, which, in turn, yields

$$u_{,11} \cdot u_{,2} + u_{,1} \cdot u_{,12} = 0 \quad \text{in } U. \tag{30}$$

Differentiating (27) with respect to x_1 and using the last equation in (28), we obtain

$$0 = u_{,112} \cdot u_{,2} + u_{,12} \cdot u_{,12} = u_{,112} \cdot u_{,2} + u_{,11} \cdot u_{,22} = \big(u_{,11} \cdot u_{,2}\big)_{,2},$$

hence, by the boundary conditions,

$$u_{,11} \cdot u_{,2} = 0 \quad \text{in } U.$$

Combining this equality with (30), we conclude that

$$0 = u_{,1} \cdot u_{,12} = \frac{1}{2}\big(u_{,1} \cdot u_{,1}\big)_{,2} \quad \text{in } U.$$

This, together with the assumption that $|y'| = 1$, yields the first equation in (26). Therefore, (26) is equivalent to (28)–(29).

Now, we observe that, if the vectors $u_{,1}$, $u_{,2}$, $u_{,11}$ are linearly independent in U, then we can solve $u_{,22}$ from (28) to obtain

$$u_{,22} = F(u_{,1}, u_{,2}, u_{,11}, u_{,12}) \quad \text{in } U,$$

where F is an analytic function of all its arguments. Owing to the analytic regularity of y and d_2, the Cauchy-Kovalevskaya theorem guarantees that the previous equation, supplemented by the initial conditions (29), always admits an analytic solution u in a neighbourhood U of $[0, \ell] \times \{0\}$. Since the vectors $u_{,1}$, $u_{,2}$, $u_{,11}$ are linearly independent on $[0, \ell] \times \{0\}$, they remain linearly independent in a small neighbourhood U' of $[0, \ell] \times \{0\}$ contained in U. Therefore, u is a solution to (28)–(29), and, in turn, of (26) in U'. □

We are now in a position to prove the main result of this section.

Proof of Theorem 4.1. **1. (Compactness)** Let (y^h) be a sequence in $W^{1,2}(\Omega; \mathbb{R}^3)$ satisfying (19). By Lemma 3.2 there exists a sequence of rotations (R^h) for which

$$\|\nabla_h y^h - R^h\|_{L^2} \le C\varepsilon_h.$$

Since R^h converge, up to subsequences, to some F weakly* in L^∞, we have that $\nabla_h y^h$ converge to F weakly in L^2. This implies that, up to additive constants, y^h converge to some y weakly in $W^{1,2}$, where $y_{,2} = y_{,3} = 0$ a.e. in Ω and $y_{,1} = Fe_1$. Since F is a weak* limit of rotations, we deduce that $|y_{,1}| \leq 1$ a.e. in Ω.

2. (Upper bound) Assume first to have a pair of analytic functions $y, d_2 : [0,\ell] \to \mathbb{R}^3$ satisfying the following conditions:

$$|y'| = |d_2| = 1, \quad y'' \neq 0, \quad \text{and} \quad y' \cdot d_2 = y'' \cdot d_2 = 0 \quad \text{in } [0,\ell]. \tag{31}$$

By Lemma 4.6, there exist some $\eta > 0$ and an analytic function $u : [0,\ell] \times [-\frac{\eta}{2}, \frac{\eta}{2}] \to \mathbb{R}^3$ such that u is an isometry on $[0,\ell] \times [-\frac{\eta}{2}, \frac{\eta}{2}]$ and satisfies

$$u(x_1, 0) = y(x_1) \quad \text{and} \quad u_{,2}(x_1, 0) = d_2(x_1) \quad \text{for every } x_1 \in [0,\ell]. \tag{32}$$

We set $n := u_{,1} \times u_{,2}$. For every $0 < h < \eta$ we consider the following recovery sequence:

$$y^h(x) := u(x_1, hx_2) + \delta_h x_3 n(x_1, hx_2)$$

for every $x \in \Omega$. By (32), it is easy to check that $y^h \to y$ strongly in $W^{1,2}(\Omega; \mathbb{R}^3)$. Moreover, we have

$$\nabla_h y^h(x) = Q(x_1, hx_2) + \mathcal{O}(\delta_h),$$

where $Q := (u_{,1}|u_{,2}|n) \in SO(3)$ and $\mathcal{O}(\delta_h)$ denotes a function such that $\mathcal{O}(\delta_h)/\delta_h$ is uniformly bounded. In particular,

$$Q^T \nabla_h y^h = I + \mathcal{O}(\delta_h),$$

so that by frame-indifference and Taylor expansion (see (4)) we obtain

$$W(\nabla_h y^h) = W(I + \mathcal{O}(\delta_h)) = \mathcal{O}(\delta_h^2).$$

By (20) we conclude that $\frac{1}{\varepsilon_h^2} W(\nabla_h y^h)$ converges to 0 uniformly, hence

$$\frac{1}{\varepsilon_h^2} \int_\Omega W(\nabla_h y^h)\, dx \longrightarrow 0.$$

Assume now that $y \in C^\infty([0,\ell]; \mathbb{R}^3)$, $|y'| = 1$, and $y'' \neq 0$ in $[0,\ell]$. For every $t \in [0,\ell]$ we set

$$n(t) := \frac{y''(t)}{|y''(t)|}, \quad b(t) := y'(t) \times n(t).$$

Since $|y'| = 1$, we have

$$R := (y' \,|\, -b \,|\, n) \in SO(3), \tag{33}$$

$$(R^T R_{,1})_{12} = -y' \cdot b' = y'' \cdot b = 0. \tag{34}$$

By Lemma 4.4, there exists a sequence of analytic functions $R^\nu : [0,\ell] \to \mathbb{R}^{3\times 3}$ such that $R^\nu \in SO(3)$, $((R^\nu)^T R^\nu_{,1})_{12} = 0$, and $R^\nu \to R$ in $C^1([0,\ell]; \mathbb{R}^{3\times 3})$ as $\nu \to \infty$.

For every $x_1 \in [0,\ell]$ we set

$$y^\nu(x_1) := y(0) + \int_0^{x_1} R^\nu(s)e_1\,ds, \qquad d_2^\nu(x_1) := R^\nu(x_1)e_2.$$

Then y^ν and d_2^ν are analytic functions and satisfy conditions (31). Indeed, as $R^\nu_{,1}$ converge to $R_{,1}$ uniformly on $[0,\ell]$ and $R_{,1}e_1 = y'' \neq 0$ in $[0,\ell]$, we can assume that $R^\nu_{,1}e_1 = (y^\nu)'' \neq 0$ in $[0,\ell]$ for ν large enough. Moreover, $y^\nu \to y$ strongly in $W^{1,2}((0,\ell);\mathbb{R}^3)$ as $\nu \to \infty$. By the previous step for every $\nu \in \mathbb{N}$ there exists a sequence of functions $(y^{\nu,h})$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ such that $y^{\nu,h} \to y^\nu$ and

$$\frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y^{\nu,h})\,dx \to 0$$

as $h \to 0$. Hence, by a standard diagonal argument, there exists a sequence $\nu_h \to 0$ such that, setting $y^h := y^{\nu_h,h}$ we have that $y^h \to y$ strongly in $W^{1,2}(\Omega;\mathbb{R}^3)$ and

$$\frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y^h)\,dx \to 0. \tag{35}$$

Finally, let $y \in W^{1,2}(\Omega;\mathbb{R}^3)$ be such that $|y_{,1}| \leq 1$, $y_{,2} = y_{,3} = 0$ a.e. in Ω. By Lemma 4.3, there exists a sequence (v_k) in $C^\infty([0,\ell];\mathbb{R}^3)$ such that $|v_k| = 1$, $v_k' \neq 0$ in $[0,\ell]$, and $v_k \rightharpoonup y'$ weakly in $L^2((0,\ell);\mathbb{R}^3)$ as $k \to \infty$. Let

$$u_k(t) := y(0) + \int_0^t v_k(s)\,ds, \qquad t \in (0,\ell),$$

so that $u_k \rightharpoonup y$ weakly in $W^{1,2}(\Omega;\mathbb{R}^3)$. By the previous step, for every k there exists a sequence of functions (y_k^h) in $W^{1,2}(\Omega;\mathbb{R}^3)$ such that $y_k^h \to u_k$ and

$$\frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y_k^h)\,dx \to 0$$

as $h \to 0$. Again by a diagonal argument we deduce that there exists a sequence $k_h \to \infty$ such that, setting $y^h := y^h_{k_h}$ we have that $y^h \rightharpoonup y$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ and (35) holds. This concludes the proof. □

5 The critical case

Throughout this section we shall consider a sequence of deformations $(y^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that

$$I^h(y^h) = \int_\Omega W(\nabla_h y^h)\,dx \leq C\varepsilon_h^2, \tag{36}$$

where

$$\lim_{h\to 0}\frac{\varepsilon_h}{\delta_h} = 1. \tag{37}$$

In this regime the class of limiting deformations is given by the following set:

$$\mathcal{A} := \big\{(y, d_2, d_3) \in W^{2,2}((0,\ell);\mathbb{R}^3) \times W^{1,2}((0,\ell);\mathbb{R}^3) \times W^{1,2}((0,\ell);\mathbb{R}^3) : \\ (y_{,1}|d_2|d_3) \in SO(3) \text{ and } y_{,1} \cdot d_{2,1} = 0 \text{ a.e. in } (0,\ell)\big\}. \quad (38)$$

Moreover, we have the following result.

Theorem 5.1 *Assume* (37). *Then the following assertions hold.*

1. (Compactness and lower bound) *Let* $(y^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *satisfy*

$$\limsup_{h\to 0} \frac{1}{\varepsilon_h^2} I^h(y^h) \le C < +\infty. \quad (39)$$

Then there exist a triple $(y, d_2, d_3) \in \mathcal{A}$ *and a subsequence (not relabeled) such that*

$$\nabla_h y^h \to (y_{,1}|d_2|d_3) \quad \text{in } L^2(\Omega;\mathbb{R}^{3\times 3}).$$

Moreover,

$$\liminf_{h\to 0} \frac{1}{\varepsilon_h^2} I^h(y^h) \ge \frac{1}{24}\int_0^\ell Q_2(y_{,1}\cdot d_{3,1}, d_2 \cdot d_{3,1})\, dx_1, \quad (40)$$

where Q_2 *is the quadratic form introduced in* (3).

2. (Upper bound) *Assume in addition that*

$$\lim_{h\to 0} \frac{h^2}{\delta_h} = 0. \quad (41)$$

Then for every $(y, d_2, d_3) \in \mathcal{A}$ *there exists a sequence* $(y^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *such that* $y^h \to y$ *in* $W^{1,2}(\Omega;\mathbb{R}^3)$, $\nabla_h y^h \to (y_{,1}|d_2|d_3)$ *in* $L^2(\Omega;\mathbb{R}^{3\times 3})$, *and*

$$\lim_{h\to 0} \frac{1}{\varepsilon_h^2} I^h(y^h) = \frac{1}{24}\int_0^\ell Q_2(y_{,1}\cdot d_{3,1}, d_2 \cdot d_{3,1})\, dx_1.$$

Proof. **1. (Compactness and lower bound)** Up to extracting a subsequence, we can assume that (y^h) satisfies (36). Thus, by Lemma 3.3 and assumption (37) we can construct a sequence $(\tilde{R}^h)$ in $C^\infty((0,\ell)\times(-1/2,1/2);\mathbb{R}^{3\times 3})$ such that for every h

$$\tilde{R}^h(x_1,x_2) \in SO(3) \text{ for every } (x_1,x_2) \in (0,\ell)\times(-1/2,1/2), \quad (42)$$

$$\|\nabla_h y^h - \tilde{R}^h\|_{L^2} \le C\varepsilon_h, \quad (43)$$

$$\|\tilde{R}^h_{,1}\|_{L^2} \le C, \quad \|\tilde{R}^h_{,2}\|_{L^2} \le Ch. \quad (44)$$

From (42) and (44) it follows that there exists $R \in W^{1,2}((0,\ell);\mathbb{R}^{3\times 3})$ such that $R(x_1) \in SO(3)$ for every $x_1 \in (0,\ell)$ and, up to subsequences, $\tilde{R}^h \rightharpoonup R$ weakly in $W^{1,2}$ and strongly in $L^2((0,\ell)\times(-1/2,1/2);\mathbb{R}^{3\times 3})$. By (43) we deduce that

$$\nabla_h y^h \to R \quad \text{in } L^2((0,\ell)\times(-\tfrac{1}{2},\tfrac{1}{2});\mathbb{R}^{3\times 3}). \quad (45)$$

This implies that $\nabla y^h \to Re_1 \otimes e_1$ strongly in $L^2((0,\ell)\times(-1/2,1/2);\mathbb{R}^{3\times3})$. Therefore, there exists $y \in W^{1,2}(\Omega;\mathbb{R}^3)$ such that, up to additive constants, $y^h \to y$ strongly in $W^{1,2}(\Omega;\mathbb{R}^3)$ and $y_{,1} = Re_1$, $y_{,2} = y_{,3} = 0$ a.e. in Ω.

Let us set $Re_k = d_k$ for $k = 2,3$. To prove that $(y, d_2, d_3) \in \mathcal{A}$ it remains to show that $y_{,1}\cdot d_{2,1} = 0$ a.e. in Ω. To this aim we remark that by the second inequality in (44) there exists $B \in L^2((0,\ell)\times(-1/2,1/2);\mathbb{R}^{3\times3})$ such that, up to subsequences,

$$\frac{1}{h}\tilde{R}^h_{,2} \rightharpoonup B \text{ weakly in } L^2((0,\ell)\times(-\tfrac{1}{2},\tfrac{1}{2});\mathbb{R}^{3\times3}). \tag{46}$$

Differentiating the identity $(\tilde{R}^h)^T\tilde{R}^h = I$, dividing by h and passing to the limit, we deduce that B satisfies

$$B^T R + R^T B = 0. \tag{47}$$

Moreover, by (43) we have

$$\begin{aligned}\Big\|\frac{1}{h}(\tilde{R}^h - \nabla_h y^h)_{,2}\Big\|_{W^{-1,2}} &= \frac{1}{h}\sup_{\|\psi\|_{W_0^{1,2}}\le 1}\Big|\int_\Omega (\tilde{R}^h - \nabla_h y^h)\,\psi_{,2}\,dx\Big| \\ &\le \frac{1}{h}\|\tilde{R}^h - \nabla_h y^h\|_{L^2} \le C\frac{\varepsilon_h}{h}.\end{aligned}$$

Combining the previous inequality with (37) and (46), we obtain

$$\frac{1}{h}(\nabla_h y^h)_{,2} \rightharpoonup B \quad \text{weakly in } W^{-1,2}(\Omega;\mathbb{R}^{3\times3}). \tag{48}$$

Now, since for every $\psi \in W_0^{1,2}(\Omega;\mathbb{R}^3)$ we have

$$\langle \frac{1}{h}y^h_{,12}, \psi\rangle = -\int_\Omega \frac{1}{h}y^h_{,2}\cdot\psi_{,1}\,dx,$$

passing to the limit and applying (45) and (48) yield

$$Be_1 = d_{2,1}.$$

By (47) we have $Be_1\cdot Re_1 = 0$, which, in view of the identity above, implies that $y_{,1}\cdot d_{2,1} = 0$ and thus, $(y, d_2, d_3) \in \mathcal{A}$.

For future reference we notice that (47) also implies that

$$Be_2\cdot d_2 = 0, \quad Be_2\cdot y_{,1} = -Be_1\cdot d_2 = -d_{2,1}\cdot d_2 = 0,$$

so that Be_2 is parallel to d_3. Since $B \in L^2((0,\ell)\times(-1/2,1/2);\mathbb{R}^{3\times3})$ and $d_3 \in L^\infty((0,\ell);\mathbb{R}^3)$, we have that $\alpha := Be_2\cdot d_3$ belongs to $L^2((0,\ell)\times(-1/2,1/2))$. Finally, again by (47) we obtain

$$\begin{aligned}&Be_3\cdot y_{,1} = -d_3\cdot d_{2,1},\\ &Be_3\cdot d_2 = -d_3\cdot Be_2 = -\alpha,\\ &Be_3\cdot d_3 = 0.\end{aligned}$$

Summarizing, we have proven that the matrix B can be represented as follows:

$$B := \big(d_{2,1}|\alpha d_3| - (d_3 \cdot d_{2,1})y_{,1} - \alpha d_2\big), \tag{49}$$

where $\alpha \in L^2((0,\ell) \times (-1/2, 1/2))$.

Let us now prove the lower bound (40). First of all we can assume that the left-hand side be finite and, possibly passing to a subsequence, that the liminf be a limit.

Let us now set

$$G^h := \frac{(\tilde{R}^h)^T \nabla_h y^h - I}{\varepsilon_h}.$$

By (43) we have that $G^h : \Omega \to \mathbb{R}^{3\times 3}$ is uniformly bounded in $L^2(\Omega; \mathbb{R}^{3\times 3})$. In particular, up to subsequences, G^h converges weakly to some G in $L^2(\Omega; \mathbb{R}^{3\times 3})$. Therefore, by Lemma 3.4 we get

$$\begin{aligned} \liminf_{h\to 0} \frac{1}{\varepsilon_h^2} \int_\Omega W(\nabla_h y^h)\, dx \quad &\geq \quad \frac{1}{2} \int_\Omega Q_3(G)\, dx = \frac{1}{2} \int_\Omega Q_3(\operatorname{sym} G)\, dx \\ &\geq \quad \frac{1}{2} \int_\Omega Q_2(G_{11}, \tfrac{1}{2}(G_{12} + G_{21}))\, dx, \end{aligned} \tag{50}$$

where the last inequality follows from the definition (3) of Q_2.

We now claim that G has the following structure:

$$G(x)e_1 = x_3 R^T d_{3,1} + G(x_1, x_2, 0)e_1, \tag{51}$$

$$G(x)e_2 = x_3 R^T B e_3 + G(x_1, x_2, 0)e_2. \tag{52}$$

Identity (51) follows by proving that $(Ge_1)_{,3} = R^T d_{3,1}$ in $W^{-1,2}(\Omega; \mathbb{R}^3)$. Indeed, by definition,

$$\tilde{R}^h G^h e_1 = \frac{1}{\varepsilon_h}(y^h_{,1} - \tilde{R}^h e_1).$$

Since $\tilde{R}^h$ does not depend on x_3, we have

$$(\tilde{R}^h G^h e_1)_{,3} = \frac{1}{\varepsilon_h} y^h_{,13} = \frac{1}{\varepsilon_h} y^h_{,31}. \tag{53}$$

Recall that $\tilde{R}^h$ converges to R weakly in $W^{1,2}((0,\ell) \times (-1/2, 1/2); \mathbb{R}^{3\times 3})$, hence strongly in $L^p((0,\ell) \times (-1/2, 1/2); \mathbb{R}^{3\times 3})$ for every $p < \infty$, while G^h converges to G weakly in $L^2(\Omega; \mathbb{R}^{3\times 3})$. Therefore, we have that $\tilde{R}^h G^h e_1 \rightharpoonup RGe_1$ in $L^2(\Omega; \mathbb{R}^3)$, hence

$$(\tilde{R}^h G^h e_1)_{,3} \rightharpoonup (RGe_1)_{,3} = R\,(Ge_1)_{,3} \quad \text{in } W^{-1,2}(\Omega; \mathbb{R}^{3\times 3}).$$

Then, passing to the limit in (53), using (45) and (37), we conclude that

$$R\,(Ge_1)_{,3} = d_{3,1},$$

which implies (51).

Identity (52) follows by proving that $(Ge_2)_{,3} = Be_3$ in $W^{-1,2}(\Omega)$. Indeed,

$$(\tilde{R}^h G^h e_2)_{,3} = \tilde{R}^h (G^h e_2)_{,3} = \frac{1}{\varepsilon_h}\big(\frac{1}{h} y^h_{,2}\big)_{,3} = \frac{1}{h\varepsilon_h} y^h_{,32}$$

Passing to the limit and using (48), we conclude that

$$R\,(Ge_2)_{,3} = Be_3,$$

which implies (52).

Combining (49), (51), (52), and the definition of R, we obtain

$$\begin{aligned}
&\int_\Omega Q_2(G_{11}, \tfrac{1}{2}(G_{12}+G_{21}))\,dx = \\
&\quad = \int_\Omega Q_2\big(x_3 y_{,1}\cdot d_{3,1} + G_{11|x_3=0}, x_3 d_2\cdot d_{3,1} + \tfrac{1}{2}(G_{12|x_3=0} + G_{21|x_3=0})\big)\,dx \\
&\quad = \int_\Omega x_3^2 Q_2\big(y_{,1}\cdot d_{3,1}, d_2\cdot d_{3,1}\big) + Q_2\big(G_{11|x_3=0}, \tfrac{1}{2}(G_{12|x_3=0} + G_{21|x_3=0})\big)\,dx \\
&\quad \geq \int_\Omega x_3^2 Q_2\big(y_{,1}\cdot d_{3,1}, d_2\cdot d_{3,1}\big)\,dx = \frac{1}{12}\int_0^\ell Q_2\big(y_{,1}\cdot d_{3,1}, d_2\cdot d_{3,1}\big)\,dx_1,
\end{aligned}$$

where the second equality follows by the fact that $y_{,1}$, d_2 and d_3 do not depend on x_3, and by the symmetry of the domain Ω. This inequality, together with (50), yields the lower bound (40).

2. (Upper bound) Let $(y, d_2, d_3) \in \mathcal{A}$ and let $R := (y_{,1}|d_2|d_3)$. Assume first that $y, d_2, d_3 \in C^2([0,\ell];\mathbb{R}^3)$. For every $h > 0$ let us consider the function

$$y^h(x) := y(x_1) + h x_2 d_2(x_1) + \delta_h x_3 d_3(x_1) - h\delta_h x_2 x_3 d_3(x_1)\cdot d_{2,1}(x_1) y_{,1}(x_1) + \beta^h(x), \tag{54}$$

with

$$\begin{aligned}
\beta^h(x) := &-h^2 \tfrac{x_2^2}{2}\beta_{22}(x_1) d_3(x_1) + h\delta_h x_2 x_3 \beta_{22}(x_1) d_2(x_1) + \\
&+\delta_h^2 x_3^2\big[\beta_{13}(x_1) y_{,1}(x_1) + \beta_{23}(x_1) d_2(x_1) + \tfrac{1}{2}\beta_{33}(x_1) d_3(x_1)\big]
\end{aligned}$$

where $(\beta_{13}(x_1), \beta_{22}(x_1), \beta_{23}(x_1), \beta_{33}(x_1))$ is the solution of the minimum problem (3) with $\alpha(x_1) = d_{3,1}\cdot y_{,1}$ and $\beta(x_1) = d_2\cdot d_{3,1}$, that is

$$Q_2(d_{3,1}\cdot y_{,1}, d_2\cdot d_{3,1}) = Q_3(B),$$

where

$$B := \begin{pmatrix} d_{3,1}\cdot y_{,1} & d_2\cdot d_{3,1} & \beta_{13} \\ d_2\cdot d_{3,1} & \beta_{22} & \beta_{23} \\ \beta_{13} & \beta_{23} & \beta_{33} \end{pmatrix}.$$

Before proceeding with the proof we note that the first term on the right of the ansatz (54) describes the deformation of the axis of the beam, the second and third terms produce a rigid rotation of the cross-sections, while the fourth term takes into account that the cross-section does not remain planar. This latter term may be interpreted as the warping of the cross-section, in fact it is proportional to $d_3 \cdot d_{2,1}$ which is a measure of torsion, see Remark 4.5.

Under the regularity assumptions on $y_{,1}$, d_2 and d_3, we have $\beta_{ij} \in C^1([0,\ell])$. Moreover,

$$\begin{aligned}\nabla_h y^h &= R + hx_2(d_{2,1}| - \beta_{22}d_3| - d_3 \cdot d_{2,1}y_{,1} + \beta_{22}d_2) \\ &\quad + \delta_h x_3(d_{3,1}| - d_3 \cdot d_{2,1}y_{,1} + \beta_{22}d_2|2\beta_{13}y_{,1} + 2\beta_{23}d_2 + \beta_{33}d_3) + \mathcal{O}(h^2),\end{aligned}$$

where $\mathcal{O}(h^2)$ denotes a function such that $\mathcal{O}(h^2)/h^2$ is uniformly bounded. It is immediate to see that $y^h \to y$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ and $\nabla_h y^h \to R = (y_{,1}|d_2|d_3)$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$.

To prove the convergence of energies we observe that from the previous computations and the fact that $d_{2,1} \cdot y_{,1} = 0$ and $d_3 \cdot d_{2,1} = -d_{3,1} \cdot d_2$, it follows that

$$\begin{aligned}R^T\nabla_h y^h &= I + hx_2 \begin{pmatrix} 0 & 0 & -d_3 \cdot d_{2,1} \\ 0 & 0 & \beta_{22} \\ d_3 \cdot d_{2,1} & -\beta_{22} & 0 \end{pmatrix} + \\ &\quad + \delta_h x_3 \begin{pmatrix} d_{3,1} \cdot y_{,1} & d_{3,1} \cdot d_2 & 2\beta_{13} \\ d_{3,1} \cdot d_2 & \beta_{22} & 2\beta_{23} \\ 0 & 0 & \beta_{33} \end{pmatrix} + \mathcal{O}(h^2),\end{aligned}$$

hence, using the formula $(I+A)^T(I+A) = I + 2\,\mathrm{sym}\,A + A^TA$,

$$(\nabla_h y^h)^T\nabla_h y^h = I + 2\delta_h x_3\,\mathrm{sym}\begin{pmatrix} d_{3,1} \cdot y_{,1} & d_{3,1} \cdot d_2 & 2\beta_{13} \\ d_{3,1} \cdot d_2 & \beta_{22} & 2\beta_{23} \\ 0 & 0 & \beta_{33} \end{pmatrix} + \mathcal{O}(h^2).$$

Therefore,

$$\sqrt{(\nabla_h y^h)^T\nabla_h y^h} = I + \delta_h x_3 B + \mathcal{O}(h^2)$$

and by frame indifference and polar decomposition

$$W(\nabla_h y^h) = W(\sqrt{(\nabla_h y^h)^T\nabla_h y^h}) = W(I + \delta_h x_3 B + \mathcal{O}(h^2)).$$

Since for h small enough the matrix $I+\delta_h x_3 B+\mathcal{O}(h^2)$ belongs to a neighborhood of I where W is of class C^2, by Taylor expansion (see (4)) we have

$$W(\nabla_h y^h) = \frac{1}{2}\delta_h^2 x_3^2 Q_3(B) + \mathcal{O}(h^4) + \mathcal{O}(h^2\delta_h) + o(\delta_h^2),$$

so that, by (37) and (41), we deduce that

$$\frac{1}{\varepsilon_h^2}W(\nabla_h y^h) \to \frac{1}{2}x_3^2 Q_3(B) \quad \text{pointwise in } \Omega,$$

and

$$\frac{1}{\varepsilon_h^2}|W(\nabla_h y^h)| \le C(|B|^2 + 1) \quad \text{in } \Omega$$

for some constant $C > 0$. We emphasize that this is the only step of the proof where we need the additional requirement (41).

By dominated convergence we conclude that

$$\lim_{h\to 0}\frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y^h)\,dx = \frac{1}{2}\int_\Omega x_3^2 Q_3(B)\,dx = \frac{1}{24}\int_0^\ell Q_2(d_{3,1}\cdot y_{,1}, d_{3,1}\cdot d_2)\,dx_1. \tag{55}$$

Consider now the general case. Let $(y, d_2, d_3) \in \mathcal{A}$ and let

$$(\beta_{13}(x_1), \beta_{22}(x_1), \beta_{23}(x_1), \beta_{33}(x_1))$$

be the solution of the minimum problem (3) with $\alpha(x_1) = d_{3,1}\cdot y_{,1}$ and $\beta(x_1) = d_2 \cdot d_{3,1}$, as above. Under the regularity assumptions on $y_{,1}$, d_2 and d_3 we have $\beta_{ij} \in L^2(0,\ell)$. We can construct, by convolution, sequences $\beta_{ij}^\nu \in C^1([0,\ell])$ such that $\beta_{ij}^\nu \to \beta_{ij}$ in $L^2(0,\ell)$.

Since $R \in W^{1,2}((0,\ell);\mathbb{R}^{3\times 3})$ and since, by the definition of R and of the set $\mathcal{A}$, we have

$$R^T R_{,1} = \begin{pmatrix} 0 & 0 & y_{,1}\cdot d_{3,1} \\ & 0 & d_2\cdot d_{3,1} \\ \text{skew} & & 0 \end{pmatrix} \in L^2(0,\ell;\mathbb{R}^{3\times 3})$$

then, by Lemma 4.4, we can find a sequence of rotations $R^\nu \in C^\infty([0,\ell];\mathbb{R}^{3\times 3})$ such that $R^\nu \to R$ in $W^{1,2}((0,\ell);\mathbb{R}^{3\times 3})$ and $(R^{\nu T} R^\nu_{,1})_{12} = 0$. Let us set

$$y^\nu(x_1) := \int_0^{x_1} R^\nu(s)e_1\,ds, \qquad d_\alpha^\nu(x_1) := R^\nu(x_1)e_\alpha \ \text{ for } \alpha = 2,3.$$

Then, y^ν, d_2^ν, $d_3^\nu \in C^\infty([0,\ell];\mathbb{R}^3)$ and $(y^\nu, d_2^\nu, d_3^\nu) \in \mathcal{A}$. Moreover, y^ν converges to y strongly in $W^{1,2}((0,\ell);\mathbb{R}^3)$, while d_2^ν and d_3^ν converge to d_2 and d_3, respectively, strongly in $W^{1,2}((0,\ell);\mathbb{R}^3)$, as $\nu \to \infty$. By the previous step for every $\nu \in \mathbb{N}$ there exists a sequence of functions $(y^{\nu,h})$ such that $y^{\nu,h} \to y^\nu$ strongly in $W^{1,2}(\Omega;\mathbb{R}^3)$, $\nabla_h y^{\nu,h} \to R^\nu = (y^\nu_{,1}|d_2^\nu|d_3^\nu)$ strongly in $L^2(\Omega;\mathbb{R}^{3\times 3})$, and

$$\frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y^{h,\nu})\,dx \to \frac{1}{24}\int_0^\ell Q_2(d^\nu_{3,1}\cdot y^\nu_{,1}, d^\nu_{3,1}\cdot d^\nu_2)\,dx_1,$$

as $h \to 0$. Hence, by a standard diagonal argument, there exists a sequence $\nu_h \to 0$ such that the sequence $y^h := y^{\nu_h,h}$ has all the required properties. $\square$

Remark 5.2 In the proof of part 2 of Theorem 5.1, we have shown that, under the assumption

$$\lim_{h\to 0}\frac{h^2}{\delta_h} = 0,$$

for every smooth triple $(y, d_2, d_3) \in \mathcal{A}$ and every smooth $\beta_{ij} : [0,\ell] \to \mathbb{R}$ it is possible to construct a sequence of smooth functions y^h such that $y^h \to y$ in

$W^{1,2}(\Omega;\mathbb{R}^3)$, $\nabla_h y^h \to (y_{,1}|d_2|d_3)$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$,

$$\lim_{h\to 0}\frac{1}{\varepsilon_h^2}\int_\Omega W(\nabla_h y^h)\,dx = \frac{1}{24}\int_0^\ell Q_3\begin{pmatrix} d_{3,1}\cdot y_{,1} & d_{3,1}\cdot d_2 & \beta_{13}\\ & \beta_{22} & \beta_{23}\\ \text{sym} & & \beta_{33}\end{pmatrix}dx_1,$$

and in addition,

$$y^h_{,11}\to y_{,11}\quad \frac{y^h_{,12}}{h}\to d_{2,1}\quad \frac{y^h_{,22}}{h^2}\to -\beta_{22}d_3\quad \text{in } L^2(\Omega;\mathbb{R}^3).$$

The aim of the present remark is simply to show that in the case

$$\lim_{h\to 0}\frac{h^2}{\delta_h} = +\infty,$$

it is impossible to find a recovery sequence satisfying all the above properties and with β_{22} an arbitrary function of x_1. To see this, suppose there exists a y^h with the above properties and note that any smooth function y^h satisfies the identity

$$-\frac{1}{2}\frac{1}{h^2}(y^h_{,1}\cdot y^h_{,1})_{,22} = -(\frac{y^h_{,22}}{h^2}\cdot y^h_{,1})_{,1} + \frac{y^h_{,22}}{h^2}\cdot y^h_{,11} - \frac{y^h_{,12}}{h}\cdot\frac{y^h_{,12}}{h}.$$

Let $\psi\in C_0^\infty(\Omega)$. Then

$$-\frac{1}{2}\frac{1}{h^2}\int_\Omega y^h_{,1}\cdot y^h_{,1}\psi_{,22}\,dx = \int_\Omega \frac{y^h_{,22}}{h^2}\cdot y^h_{,1}\psi_{,1} + (\frac{y^h_{,22}}{h^2}\cdot y^h_{,11} - \frac{y^h_{,12}}{h}\cdot\frac{y^h_{,12}}{h})\psi\,dx. \quad (56)$$

The left-hand side of (56) converges to zero, since

$$\begin{aligned}\frac{1}{h^2}\Big|\int_\Omega y^h_{,1}\cdot y^h_{,1}\psi_{,22}\,dx\Big| &= \frac{1}{h^2}\Big|\int_\Omega [(y^h_{,1}-\tilde{R}^h e_1)\cdot y^h_{,1} + \tilde{R}^h e_1\cdot(y^h_{,1}-\tilde{R}^h e_1)\\ &\qquad +\tilde{R}^h e_1\cdot\tilde{R}^h e_1]\psi_{,22}\,dx\Big|\\ &= \frac{1}{h^2}\Big|\int_\Omega (y^h_{,1}-\tilde{R}^h e_1)\cdot(y^h_{,1}+\tilde{R}^h e_1)\psi_{,22}\,dx\Big|\\ &\le \frac{1}{h^2}\|(y^h_{,1}-\tilde{R}^h e_1)\|_{L^2(\Omega)}\|(y^h_{,1}+\tilde{R}^h e_1)\psi_{,22}\|_{L^2(\Omega)}\\ &\le C\frac{\delta_h}{h^2},\end{aligned}$$

where we have applied Lemma 3.3 to the sequence (y^h).

Thus passing to the limit in (56), under the assumed convergence, we find

$$0 = \int_\Omega -\beta_{22}d_3\cdot y_{,1}\psi_{,1} + (-\beta_{22}d_3\cdot y_{,11} - d_{2,1}\cdot d_{2,1})\psi\,dx.$$

Hence, taking into account that $d_{2,1} = (d_{2,1}\cdot d_3)d_3$, we deduce

$$0 = (y_{,1}\cdot d_{3,1})\beta_{22} - (d_{2,1}\cdot d_3)^2,$$

which implies that β_{22} can not be a generic function. Hence the contradiction.

6 Convergence of minimizers

In this section we consider a sequence of forces and we characterize the asymptotic behaviour, as $h \to 0$, of minimizers (or almost minimizers) of the total energy. This is made precise in the following two theorems.

Theorem 6.1 (Subcritical case) *Let* $\varepsilon_h \to 0$ *and*

$$\lim_{h\to 0} \frac{\delta_h}{\varepsilon_h} = 0. \tag{57}$$

Let $f^h, f \in L^2(\Omega;\mathbb{R}^3)$ *be such that*

$$\int_\Omega f^h(x)\,dx = 0 \qquad \text{and} \qquad \frac{1}{\varepsilon_h^2} f^h \rightharpoonup f \ \text{ weakly in } L^2(\Omega;\mathbb{R}^3). \tag{58}$$

Let

$$J^h(y) := I^h(y) - \int_\Omega f^h \cdot y\,dx$$

for $y \in W^{1,2}(\Omega;\mathbb{R}^3)$.

Then $|\inf J^h| \le C\varepsilon_h^2$. *Moreover, if* $(y^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *is a minimizing sequence of* $\frac{1}{\varepsilon_h^2} J^h$ *in the following sense*

$$\lim_{h\to 0}\Big(\frac{1}{\varepsilon_h^2} J^h(y^h) - \inf \frac{1}{\varepsilon_h^2} J^h\Big) = 0, \tag{59}$$

then there exist some constants $c^h \in \mathbb{R}^3$ *and a function* $\bar y \in W^{1,2}(\Omega;\mathbb{R}^3)$ *such that, up to subsequences,* $y^h - c^h \rightharpoonup \bar y$ *weakly in* $W^{1,2}(\Omega;\mathbb{R}^3)$. *The limit* $\bar y$ *satisfies* $|\bar y_{,1}| \le 1$, $\bar y_{,2} = \bar y_{,3} = 0$ *a.e. in* Ω *and minimizes the functional*

$$J_{sub}(y) = -\int_0^\ell \bar f \cdot y\,dx_1$$

among all $y \in W^{1,2}((0,\ell);\mathbb{R}^3)$ *with* $|y'| \le 1$. *Here we have set* $\bar f(x_1) := \int_{(-\frac12,\frac12)^2} f(x)\,dx_2dx_3$. *Furthermore, we have*

$$\lim_{h\to 0} \frac{1}{\varepsilon_h^2} \inf J^h = \lim_{h\to 0} \frac{1}{\varepsilon_h^2} J^h(y^h) = J_{sub}(\bar y) = \min J_{sub}.$$

Proof. For further reference we divide the proof into two steps.
Step 1. Consider the identity deformation $p_h(x) := (x_1, hx_2, \delta_h x_3)$. Then by Hölder inequality and (58) we have

$$\inf J^h \le J^h(p_h) = -\int_\Omega f^h \cdot p_h\,dx \le C\|f^h\|_{L^2} \le C\varepsilon_h^2. \tag{60}$$

Let now (y^h) be a minimizing sequence of $\frac{1}{\varepsilon_h^2}J^h$ in the sense of (59), and let us denote by c^h the average of y^h on Ω. Using (58), (59), (60), and Poincaré-Wirtinger inequality we obtain

$$\begin{aligned} I^h(y^h) &= J^h(y^h) + \int_\Omega f^h \cdot (y^h - c^h)\, dx \\ &\le C\varepsilon_h^2 + \varepsilon_h^2 \|y^h - c^h\|_{L^2} \le C\varepsilon_h^2 + C\varepsilon_h^2 \|\nabla y^h\|_{L^2}. \end{aligned} \tag{61}$$

From the assumptions on W it follows that $W(F) \ge C|F|^2 - C'$ for every $F \in \mathbb{R}^{3\times3}$ and for some constant $C, C' > 0$, so that

$$I^h(y^h) \ge C\|\nabla_h y^h\|_{L^2}^2 - C'|\Omega| \ge C\|\nabla y^h\|_{L^2}^2 - C'|\Omega|.$$

Combining the two previous estimates we deduce that the sequence (∇y^h) is uniformly bounded in $L^2(\Omega;\mathbb{R}^{3\times3})$. In particular, by (61) this implies that

$$I^h(y^h) \le C\varepsilon_h^2. \tag{62}$$

Moreover, using again (58) and Poincaré-Wirtinger inequality, we obtain

$$J^h(y^h) \ge -\int_\Omega f^h \cdot (y^h - c^h)\, dx \ge -C\varepsilon_h^2 \|\nabla y^h\|_{L^2} \ge -C\varepsilon_h^2.$$

Since (y^h) is a minimizing sequence, the last inequality together with (60) implies that $|\inf J^h| \le C\varepsilon_h^2$.
STEP 2. Assume now (57). By (62) and by Theorem 4.1, part 1, there exist some constants $c^h \in \mathbb{R}^3$ and a function $\bar{y} \in W^{1,2}(\Omega;\mathbb{R}^3)$ such that $|\bar{y}_{,1}| \le 1$, $\bar{y}_{,2} = \bar{y}_{,3} = 0$ a.e. in Ω and $y^h - c^h \rightharpoonup \bar{y}$ in $W^{1,2}(\Omega;\mathbb{R}^3)$. In particular, $y^h - c^h \to \bar{y}$ in $L^2(\Omega;\mathbb{R}^3)$. Using (58) we deduce that

$$\begin{aligned} \liminf_{h\to0} \frac{1}{\varepsilon_h^2} J^h(y^h) &\ge \liminf_{h\to0} \Big(-\int_\Omega \frac{1}{\varepsilon_h^2} f^h \cdot (y^h - c^h)\, dx \Big) \\ &= -\int_\Omega f \cdot \bar{y}\, dx = J_{sub}(\bar{y}). \end{aligned} \tag{63}$$

Let now $y \in W^{1,2}((0,\ell);\mathbb{R}^3)$ with $|y'| \le 1$. By Theorem 4.1, part 2, there exists a sequence $(\hat{y}^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that $\hat{y}^h \rightharpoonup y$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ and

$$\lim_{h\to0} \frac{1}{\varepsilon_h^2} I^h(\hat{y}^h) = 0.$$

This implies that

$$\begin{aligned} \limsup_{h\to0} \frac{1}{\varepsilon_h^2} J^h(y^h) &= \limsup_{h\to0} \Big(\frac{1}{\varepsilon_h^2} \inf J^h \Big) \le \limsup_{h\to0} \frac{1}{\varepsilon_h^2} J^h(\hat{y}^h) \\ &= \limsup_{h\to0} \Big(-\int_\Omega \frac{1}{\varepsilon_h^2} f^h \cdot \hat{y}^h\, dx \Big) = J_{sub}(y). \end{aligned} \tag{64}$$

Combining (63) and (64) we have the thesis. □

In the critical case we may consider more general loads, see Remark 6.3 below.

Theorem 6.2 (Critical case) *Let*

$$\lim_{h\to 0}\frac{\delta_h}{\varepsilon_h}=1, \tag{65}$$

and suppose in addition that (41) *is satisfied. Let* $F^h, F \in L^2(\Omega;\mathbb{R}^{3\times 3})$ *be such that*

$$\frac{1}{\varepsilon_h^2}F^h \rightharpoonup F \text{ weakly in } L^2(\Omega;\mathbb{R}^{3\times 3}). \tag{66}$$

Let

$$J^h(y) := I^h(y) - \int_\Omega F^h\cdot\nabla_h y\,dx$$

for $y \in W^{1,2}(\Omega;\mathbb{R}^3)$.

Then $|\inf J^h| \le C\varepsilon_h^2$. *Moreover, if* $(y^h) \subset W^{1,2}(\Omega;\mathbb{R}^3)$ *is a minimizing sequence of* $\frac{1}{\varepsilon_h^2}J^h$ *in the sense of* (59)*, then there exist some constants* $c^h \in \mathbb{R}^3$ *and a triple* $(\bar y, \bar d_2, \bar d_3) \in \mathcal{A}$ *such that, up to subsequences,* $y^h - c^h \to \bar y$ *strongly in* $W^{1,2}(\Omega;\mathbb{R}^3)$ *and* $\nabla_h y^h \to (\bar y_{,1}|\bar d_2|\bar d_3)$ *in* $L^2(\Omega;\mathbb{R}^{3\times 3})$. *Moreover, the limit triple* $(\bar y, \bar d_2, \bar d_3)$ *minimizes the functional*

$$J_{crit}(y,d_2,d_3) = \frac{1}{24}\int_0^\ell Q_2(y_{,1}\cdot d_{3,1}, d_2\cdot d_{3,1})\,dx_1 - \int_0^\ell \bar F\cdot(y_{,1}|d_2|d_3)\,dx_1$$

among all triples $(y,d_2,d_3)\in\mathcal{A}$*, where we have set* $\bar F(x_1) := \int_{(-\frac12,\frac12)^2} F(x)\,dx_2dx_3$. *Furthermore, we have*

$$\lim_{h\to 0}\frac{1}{\varepsilon_h^2}\inf J^h = \lim_{h\to 0}\frac{1}{\varepsilon_h^2}J^h(y^h) = J_{crit}(\bar y,\bar d_2,\bar d_3) = \min J_{crit}.$$

Proof. As in the case of Theorem 6.1 the proof can be divided into two steps.
Step 1. This is quite similar to that of Theorem 6.1.
Step 2. Assume (41) and (65). By (62) and by Theorem 5.1, part 1, there exist some constants $c^h \in \mathbb{R}^3$ and a triple $(\bar y, \bar d_2, \bar d_3)\in\mathcal{A}$ such that, up to subsequences, $y^h - c^h \to \bar y$ strongly in $W^{1,2}(\Omega;\mathbb{R}^3)$ and $\nabla_h y^h \to (\bar y_{,1}|\bar d_2|\bar d_3)$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$. Moreover, by (40) and (66) we deduce that

$$\liminf_{h\to 0}\frac{1}{\varepsilon_h^2}J^h(y^h) \ge J_{crit}(\bar y,\bar d_2,\bar d_3). \tag{67}$$

Let now $(y,d_2,d_3)\in\mathcal{A}$. By Theorem 5.1, part 2, there exists a sequence $(\hat y^h)\subset W^{1,2}(\Omega;\mathbb{R}^3)$ such that $\hat y^h \to y$ in $W^{1,2}(\Omega;\mathbb{R}^3)$ and

$$\lim_{h\to 0}\frac{1}{\varepsilon_h^2}I^h(\hat y^h) = \frac{1}{24}\int_0^\ell Q_2(y_{,1}\cdot d_{3,1}, d_2\cdot d_{3,1})\,dx_1.$$

This implies that

$$\begin{aligned}\limsup_{h\to 0}\frac{1}{\varepsilon_h^2}J^h(y^h) &= \limsup_{h\to 0}\left(\frac{1}{\varepsilon_h^2}\inf J^h\right)\\ &\le \limsup_{h\to 0}\frac{1}{\varepsilon_h^2}J^h(\hat y^h) = J_{crit}(y,d_2,d_3). \end{aligned}\tag{68}$$

Combining (67) and (68) we have the thesis. □

Remark 6.3 Let $S^h := e_1 \otimes e_1 + 1/h e_2 \otimes e_2 + 1/\delta_h e_3 \otimes e_3$. If F^h besides being in $L^2(\Omega;\mathbb{R}^{3\times 3})$ is slightly more regular then

$$\int_\Omega F^h \cdot \nabla_h y\, dx = \int_\Omega f^h \cdot y\, dx + \int_{\partial\Omega} s^h \cdot y\, dx, \tag{69}$$

for any $y \in W^{1,2}(\Omega;\mathbb{R}^3)$, where the body and surface forces f^h and s^h are given by

$$f^h := -\mathrm{div} F^h S^h, \qquad s^h := F^h S^h \nu,$$

with ν the exterior normal to the boundary of Ω.

Conversely, for any $f^h \in L^2(\Omega;\mathbb{R}^3)$ and $s^h \in L^2(\partial\Omega;\mathbb{R}^3)$ such that

$$\int_\Omega f^h\, dx + \int_{\partial\Omega} s^h\, dx = 0,$$

it is always possible to find an $F^h \in L^2(\Omega;\mathbb{R}^{3\times 3})$ for which (69) is satisfied.

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References

[1] E. Acerbi, G. Buttazzo, D. Percivale, A variational definition of the strain energy for an elastic string, *J. Elasticity* **25** (1991), 137–148.

[2] S.S. Antman, *Nonlinear Problems of Elasticity*. Springer, New York, 2005.

[3] G. Anzellotti, S. Baldo, D. Percivale, Dimension reduction in variational problems, asymptotic development in Γ-convergence and thin structures in elasticity, *Asymptotic Anal.* **9** (1994), 61–100.

[4] S. Conti, *Low-energy deformations of thin elastic sheets: isometric embeddings and branching patterns*. Habilitation thesis, Universität Leipzig, 2003.

[5] S. Conti, F. Maggi, Confining thin elastic sheets and folding paper, *Arch. Ration. Mech. Anal.* **187** (2008), 1–48.

[6] G. Dal Maso, *An introduction to Γ-convergence.* Birkhäuser, Boston, 1993.

[7] L. Freddi, A. Morassi, R. Paroni, Thin-walled beams: the case of the rectangular cross-section, *J. Elasticity* **76** (2004), 45–66.

[8] L. Freddi, A. Morassi, R. Paroni, Thin-walled beams: a derivation of Vlassov theory via Γ-convergence, *J. Elasticity* **86** (2007), 263–296.

[9] L. Freddi, F. Murat, R. Paroni, Anisotropic inhomogeneous rectangular thin-walled beams, *SIAM J. Math. Anal.* **40** (2009), 1923–1951.

[10] G. Friesecke, R.D. James, S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity, *Comm. Pure Appl. Math.* **55** (2002), 1461–1506.

[11] G. Friesecke, R.D. James, S. Müller, A hierarchy of plate models derived from nonlinear elasticity by gamma-convergence., *Arch. Ration. Mech. Anal.* **180** (2006), 183–236.

[12] M.E. Gurtin, *An introduction to continuum mechanics.* Mathematics in Science and Engineering, **158**. Academic Press, New York-London, 1981.

[13] Q. Han, J.-X. Hong, *Isometric embedding of Riemannian manifolds in Euclidean spaces.* Mathematical Surveys and Monographs, **130**. American Mathematical Society, Providence, RI, 2006.

[14] A. Mielke, Saint-Venant's problem and semi-inverse solutions in nonlinear elasticity, *A*rch. Rational Mech. Anal. **102** (1988), 205–229.

[15] M.G. Mora, S. Müller, Derivation of the nonlinear bending-torsion theory for inextensible rods by Γ-convergence, *Calc. Var.* **18** (2003), 287–305.

[16] M.G. Mora, S. Müller, A nonlinear model for inextensible rods as a low energy Gamma-limit of three-dimensional nonlinear elasticity, *Ann. Inst. H. Poincaré Anal. Nonlin.* **21** (2004), 271–293.

[17] A.C. Pipkin, Energy minimization for nets with slack, *Quart. Appl. Math.* **44** (1986), 249–253.

[18] L. Scardia, The nonlinear bending-torsion theory for curved rods as Γ-limit of three-dimensional elasticity, *Asymptot. Anal.* **47** (2006), 317–343.

[19] L. Scardia, Asymptotic models for curved rods derived from nonlinear elasticity by Γ-convergence, *Proc. Roy. Soc. Edinburgh Sect. A* **139** (2009), 1037–1070.

[20] M. Spivak, *A comprehensive introduction to differential geometry.* Vol. V. Second edition. Publish or Perish, Inc., Wilmington, Del., 1979.