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Quasistatic delamination models for Kirchhoff-Love plates

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Abstract. A quasistatic rate-independent brittle delamination problem and also an adhesive unilateral contact problem is considered on a prescribed normally-positioned surface in a plate with a finite thickness. By letting the thickness of the plate go to zero, two quasistatic rate-independent crack models with prescribed path for Kirchhoff-Love plates are obtained as limit of these quasistatic processes.

Key Words. Brittle delamination, adhesive contact, rate-independent processes, dimension reduction, Γ-convergence, Kirchhoff-Love plates.

AMS Subject Classification: 49J45, 49S05, 74K20, 74R10.

1 Introduction, notation, basic concepts

This paper aims to study crack propagation in two-dimensional plates and to rigorously derive two models as a limit from the three-dimensional theory of brittle delamination and adhesive contact. We confine ourselves to *small strains* and a-priori *prescribed surface* where delamination may occur. Furthermore, we restrict our attention to *quasistatic* unidirectional (i.e. healing is impossible) *rate-independent delamination.* We consider a *Signorini contact*, which is important to prevent (unphysical) delamination by mere compression. We further confine ourselves to *Griffith-type delamination* on such a prescribed surface which is positioned in a normal direction to the mid-plane of the plate. In particular, we do not distinguish various modes of delamination (like pulling vs. shearing) and count that always the same prescribed energy is needed for the delamination. The variational dimension reduction process leads to a *Kirchhoff-Love* model for the plate. A similar model has been obtained also in [13] as a limit of a dimension reduction problem involving a Barenblatt-like cohesive crack surface energy. For a static delamination on a generally-positioned delamination surface in Kirchhoff-Love plates, we refer to [15, 16].

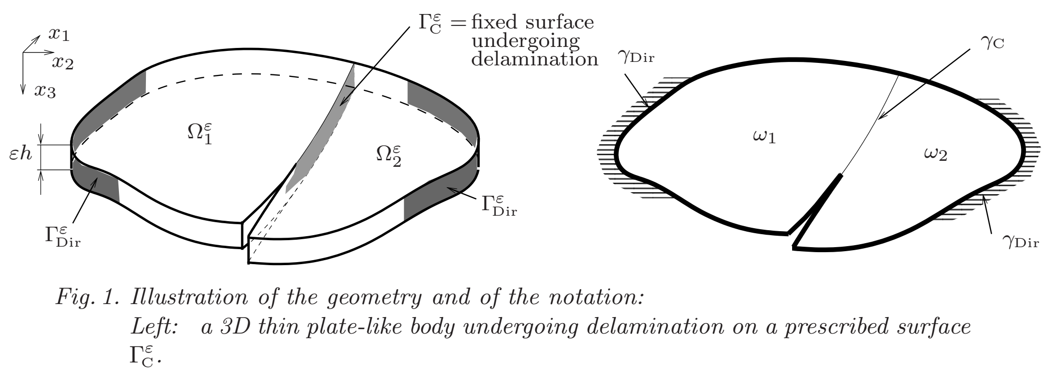

Fig. 1. Illustration of the geometry and of the notation:
Left: a 3D thin plate-like body undergoing delamination on a prescribed surface Γ_C^ε.
Right: a 2D plate obtained for $\varepsilon \to 0$ *undergoing crack on the curve* γ_C.

For notational simplicity, we confine ourselves to only one delamination surface, dividing a 3-dimensional elastic body into two parts occupying respectively the domains Ω_1^ε and Ω_2^ε connected mutually by a contact boundary Γ_C^ε, cf. Figure 1(left). To simplify mathematical consideration without restricting substantially the possible applications, we assume both parts Ω_1^ε and Ω_2^ε to be fixed by Dirichlet boundary conditions on some parts of the side boundary not directly connected with $\Gamma_{\rm C}^\varepsilon$.

We consider a rather special "cylindrical" case, i.e., in particular the boundaries $\Gamma_{\rm C}^\varepsilon$ and $\Gamma_{\rm Dir}^\varepsilon$ are positioned vertically. More precisely, we consider a bounded open Lipschitz subset ω of $\mathbb{R}^2$ and its decomposition $\omega = \omega_1 \cup \gamma_{\rm C} \cup \omega_2$, where ω_1 and ω_2 are two disjoint open Lipschitz connected subsets with a non-void simply connected common boundary $\gamma_{\rm C}$. For $i = 1, 2$, assume that hard-device loads (i.e. Dirichlet boundary conditions) will act on a nonvanishing part $\gamma_{\rm Dir}^i$ of $\partial\omega_i$ far from the delamination surface, i.e.

$$\mathscr{H}^1(\gamma_{\rm Dir}^i) > 0, \qquad \overline{\gamma_{\rm C}} \cap \overline{\gamma_{\rm Dir}^i} = \emptyset,$$

where $\mathscr{H}^1$ denotes the 1-dimensional Hausdorff measure. Set $\gamma_{\rm Dir} := \gamma_{\rm Dir}^1 \cup \gamma_{\rm Dir}^2$, and define

$$\Omega_i^\varepsilon := \omega_i \times \Big(-\frac{\varepsilon h}{2}, \frac{\varepsilon h}{2}\Big), \quad \Gamma_{\rm C}^\varepsilon := \gamma_{\rm C} \times \Big(-\frac{\varepsilon h}{2}, \frac{\varepsilon h}{2}\Big), \quad \Gamma_{\rm Dir}^\varepsilon := \gamma_{\rm Dir} \times \Big(-\frac{\varepsilon h}{2}, \frac{\varepsilon h}{2}\Big). \tag{1.1}$$

Hence the decomposition $\Omega^\varepsilon = \Omega_1^\varepsilon \cup \Gamma_{\rm C}^\varepsilon \cup \Omega_2^\varepsilon$ holds. The constant parameter h is kept to clarify the role played by the thickness of the body in the limit problem.

This will allow for the dimensional reduction by letting the aspect ratio ε go to 0. The geometry of the resulted plate is then depicted in Figure 1(right).

Thoroughout the whole article, we will use a rather special general framework, namely that the driving force has a potential, denoted by $\mathcal{E}(t,\cdot,\cdot)$, and that the dissipation rate, denoted by $\mathcal{R}$, is independent of the state itself. Then the quasistatic evolution will be determined by the stored energy $\mathcal{E} : [0,T] \times \mathcal{U} \times \mathcal{Z} \to \mathbb{R} \cup \{+\infty\}$ and the potential of dissipative force $\mathcal{R} : \mathcal{Z} \to [0, +\infty]$. The quasistatic evolution we have in mind is governed by the following system of *doubly nonlinear* degenerate *parabolic/elliptic variational inclusions*:

$$\partial_u \mathcal{E}(t, u, z) \ni 0 \quad \text{and} \quad \partial\mathcal{R}\big(\dot z\big) + \partial_z \mathcal{E}(t, u, z) \ni 0, \tag{1.2}$$

where "∂" refers to a (partial) subdifferential, relying on that $\mathcal{R}(\cdot)$, $\mathcal{E}(t,\cdot,z)$, and $\mathcal{E}(t,u,\cdot)$ are convex functionals, which will be indeed always satisfied here. An important assumption is that $\mathcal{R}$ is degree-1 positively homogeneous, which implies that (and even is equivalent to) the dissipation rate is just the potential of a dissipative force.

Also this implies that, if $\mathcal{E}(t,\cdot,\cdot)$ is convex, the conventional weak solutions are basically equivalent (under mild additional temporal regularity assumptions) to so-called *energetic solutions* of the rate-independent system $(\mathcal{U}\times\mathcal{Z}, \mathcal{E}, \mathcal{R})$ with the initial conditions

$$u(0) = u^0 \quad \text{and} \quad z(0) = z^0. \tag{1.3}$$

Definition 1.1 (Energetic solution.) *The process $q = (u, z) : [0,T] \to \mathcal{Q} := \mathcal{U} \times \mathcal{Z}$ is called an energetic solution of the initial-value problem given by $(\mathcal{U}\times\mathcal{Z}, \mathcal{E}, \mathcal{R}, u^0, z^0)$ if, beside* (1.3), $t \mapsto \partial_t\mathcal{E}(t, q(t)) \in L^1((0,T))$, *if for all $t \in [0,T]$ we have $\mathcal{E}(t, q(t)) < +\infty$ and if the global* stability *inequality* (1.4a) *and the global* energy balance (1.4b) *are satisfied for all $t \in [0,T]$:*

$$\forall \check q = (\check u, \check z) \in \mathcal{Q}: \qquad \mathcal{E}(t, q(t)) \le \mathcal{E}(t, \check q) + \mathcal{R}(\check z - z(t)), \tag{1.4a}$$

$$\mathcal{E}(t, q(t)) + \mathrm{Diss}_{\mathcal{R}}(z, [0,t]) = \mathcal{E}(0, q(0)) + \int_0^t \partial_s \mathcal{E}(s, q(s))\,\mathrm{d}s \tag{1.4b}$$

with $\mathrm{Diss}_{\mathcal{R}}(z,[0,t]) := \sup \sum_{j=1}^N \mathcal{R}(z(t_j) - z(t_{j-1}))$, *where the supremum is taken over all partitions of* $[0,t]$.

If $\mathcal{E}(t,\cdot,\cdot)$ is separately convex but nonconvex, as in this work, then (1.2) and (1.4) are no longer equivalent. The energetic formulation (1.4) then represents a proper generalized formulation based on a minimum-energy principle competing with the maximum-dissipation principle or rather with Levitas' realizability principle [24], cf. [26, 31, 32]. Another justification is by a minimum dissipation-potential principle, saying that, at any time t, the rate $\dot z$ minimizes

$$\dot z \mapsto \mathscr{L}(t, u, z, \dot z) := \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E} + \mathcal{R} = \mathcal{E}_t'(t, u, z) + \big\langle \mathcal{E}_z'(t, u, z), \dot z \big\rangle + \mathcal{R}(\dot z), \tag{1.5}$$

where the equality relies on the fact that u minimizes $\mathcal{E}(t,\cdot,z)$, as stated in [10]. The main advantages of the energetic-solution concept are that it is derivative-free, i.e. there is no $\partial_u\mathcal{E}$, nor $\partial_z\mathcal{E}$, neither $\dot{z}$ in Definition 1.1, and that it can be handled by Calculus of Variations techniques (in particular variational convergence, as shown in [30] and as exploited also here) as well as strictly linked with direct numerical methods, as shown in [28]. For its application to delamination, namely to the problem denoted below by $(\mathcal{U}\times\mathcal{Z}, \mathsf{E}_{\varepsilon,\kappa}, \mathsf{R}, \mathsf{q}^0_\kappa)$, with numerical implementation and computational simulations we refer to [20, 34]. Roughly speaking, energetic solutions tempt to evolve as soon as it is energetically convenient. This may, however, not be exactly always in full agreement with the response of real systems involving some other rate-dependent phenomena. Therefore, in spite of these theoretical and computational arguments supporting the energetic-solution concept, there are also some other concepts of solutions that are sometimes applicable and succesfully competing with energetic solutions, cf. also [27] for a comparison with other notions in general and e.g. [4, 17, 18, 19, 21, 33] in the context of crack propagation.

Besides the mentioned Griffith-type model, we shall be concerned with adhesive-type (or sometimes also called elastic/brittle) models and how they approximate the Griffith-type model. As a consequence, we will use several stored energies. In particular, $\mathsf{E}_{\varepsilon,\kappa}$ and E_ε will denote the energies in the adhesive contact and in the brittle delamination models, respectively, while $\mathsf{E}_{0,\kappa}$ and E_0 will denote the limit 2D models described in Section 4 and rewritten in terms of Kirchhoff-Love displacements in the concluding Section 6. For simplicity, the rigidity of the adhesive will be described by a scalar parameter κ, although more realistic delamination models often involve an anisotropic behaviour. Our results can simply be summarized in the following diagram where the arrows have the meaning of convergence of the energetic solutions in the sense precised by the theorems aside the arrows

$$\begin{array}{ccc} (\mathcal{U}\times\mathcal{Z}, \mathsf{E}_{\varepsilon,\kappa}, \mathsf{R}, \mathsf{q}^0_\kappa) & \xrightarrow[\text{Theorem 5.5}]{\varepsilon\to0^+} & (\mathcal{U}\times\mathcal{Z}, \mathsf{E}_{0,\kappa}, \mathsf{R}, \mathsf{q}^0_\kappa) \\ \Big\downarrow {\scriptstyle \kappa\to+\infty \atop \text{Proposition 5.7}} & & \Big\downarrow {\scriptstyle \kappa\to+\infty \atop \text{Proposition 5.7}} \\ (\mathcal{U}\times\mathcal{Z}, \mathsf{E}_{\varepsilon}, \mathsf{R}, \mathsf{q}^0) & \xrightarrow[\text{Theorem 5.6}]{\varepsilon\to0^+} & (\mathcal{U}\times\mathcal{Z}, \mathsf{E}_{0}, \mathsf{R}, \mathsf{q}^0) \end{array} \tag{1.6}$$

We will use the standard notation as far as the function spaces concerns: C^k for functions with k-th derivatives continuous, L^p for Lebesgue spaces and $W^{k,p}$ for Sobolev spaces. With a little abuse of notation, and since this is a common practice and does not give rise to any mistake, we use to call "sequences" even those families indicized by a continuous parameter $\varepsilon \in (0,1)$.

2 The brittle delamination or the adhesive unilateral contact in 3D bodies

In this section we present two models for delamination in 3D bodies. The first is *brittle delamination* based essentially on the *Griffith concept* [9], cf. also Remark 2.2 below. There the two parts of the body can be delaminated just by a phenomenologically prescribed energy density a_ε (in physical units J/m^2). For later purposes, we indicate the dependence on the thickness parameter ε, even if it will be considered fixed thoroughout this whole Section 2. To admit an arbitrary topology of the 2D delaminated area, we use a delamination parameter $z : \Gamma_{\rm C} \to [0,1]$ representing the fraction of fixed adhesive: $z(x) = 0$ means complete delamination, $z(x) = 1$ means perfect integrity and, for instance, $z(x) = \frac{1}{2}$ means that 50% of the adhesive is debonded at $x \in \Gamma_{\rm C}$. In fact, this idea is essentially "borrowed" from the usual concept of adhesive contact, e.g. [14, 20, 36]. This model, see [35], is determined by the stored and the dissipation energies

$$\widetilde{\mathcal{E}}_\varepsilon(t,u,z) = \begin{cases} \displaystyle\frac{1}{2}\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} \mathbb{C}e(u){:}e(u) - 2f^\varepsilon(t){\cdot}u\,\mathrm{d}x & \text{if } (u,z)\in\mathcal{A}_\varepsilon(\psi^\varepsilon(t)), \\ +\infty & \text{else,} \end{cases} \tag{2.1}$$

$$\mathcal{R}_\varepsilon(\dot z) := \begin{cases} \displaystyle\int_{\Gamma_{\rm C}^\varepsilon} a_\varepsilon|\dot z|\,\mathrm{d}\mathscr{H}^2 & \text{if } \dot z \le 0 \text{ on } \Gamma_{\rm C}^\varepsilon, \\ +\infty & \text{else,} \end{cases} \tag{2.2}$$

where $e(u)_{ij} = \frac{1}{2}(\partial_{x_i}u_j + \partial_{x_j}u_i)$ is the symmetric part of the gradient of u, $[\![\cdot]\!]_{\Gamma_{\rm C}^\varepsilon}$ denotes the jump across the surface $\Gamma_{\rm C}^\varepsilon$, ν is the normal to $\Gamma_{\rm C}^\varepsilon$ and

$$\mathcal{A}_\varepsilon(\psi) = \big\{(u,z) \in W^{1,2}(\Omega_1^\varepsilon \cup \Omega_2^\varepsilon;\mathbb{R}^3) \times L^\infty(\Gamma_{\rm C}^\varepsilon) \ : \ u = \psi \text{ on } \Gamma_{\rm Dir}^\varepsilon,$$
$$0 \le z \le 1, \ \ [\![u{\cdot}\nu]\!]_{\Gamma_{\rm C}^\varepsilon} \ge 0, \text{ and } z[\![u]\!]_{\Gamma_{\rm C}^\varepsilon} = 0 \text{ on } \Gamma_{\rm C}^\varepsilon\big\}.$$

The load by body forces $f^\varepsilon \in C^1([0,T];L^2(\Omega^\varepsilon;\mathbb{R}^3))$ and the boundary displacement $\psi^\varepsilon \in C^1([0,T];H^{1/2}(\Gamma_{\rm Dir}^\varepsilon;\mathbb{R}^3))$ will be precised better in the next section.

We assume that $\mathbb{C}$ is a fourth-order positive-definite tensor, i.e.

$$\mathbb{C}e{:}e \ge c|e|^2 \tag{2.3}$$

for every symmetric matrix $e \in \mathbb{R}^{3\times 3}_{\rm sym}$ and with some $c > 0$ and that

$$a_\varepsilon(x) \ge a_{\varepsilon,\min}$$

for a suitable constant $a_{\varepsilon,\min} > 0$ (depending on ε, but not on x) and $\mathscr{H}^2$-a.e. $x \in \Gamma_{\rm C}^\varepsilon$. Moreover, we assume the following usual symmetry properties

$$\mathbb{C}_{ijkl} = \mathbb{C}_{ijlk} = \mathbb{C}_{klij}, \qquad i,j,k,l = 1,2,3, \tag{2.4}$$

and that the material has monoclinic symmetry with respect to the (x_1,x_2)-plane, which implies

$$\mathbb{C}_{\alpha\beta\gamma 3} = \mathbb{C}_{\alpha 333} = 0, \qquad \alpha,\beta,\gamma = 1,2. \tag{2.5}$$

We restrict to monoclinic symmetry for simplicity. In the absence of delamination, under this assumption, the limit problem decouples into two separate problems: one consisting in the in-plane equilibrium equations and the other for the out-of-plane equations.

The other model, called *adhesive contact* or sometimes in the engineering literature *elastic-brittle contact*, relies on the idea that the two parts of the body are glued together with an elastic adhesive which can be delaminated. Delamination is phenomenologically described by an energy density as used and analysed in [20]. We consider, rather for simplicity, the elastic response in the adhesive described by a function κQ^ε, where κ is a scalar which measures the rigidity of the adhesive, and

$$Q^\varepsilon\big([\![u]\!]_{\Gamma_{\rm C}^\varepsilon}\big) := \big|[\![u_1]\!]_{\Gamma_{\rm C}^\varepsilon}\big|^2 + \big|[\![u_2]\!]_{\Gamma_{\rm C}^\varepsilon}\big|^2 + \varepsilon^2\big|[\![u_3]\!]_{\Gamma_{\rm C}^\varepsilon}\big|^2$$

is a quadratic form which modulates differently the components of the jump of u. The different scalings of the terms in Q^ε take into account the different rigidity in the in-plane and in the out-of-plane directions of the cylinder of height εh. In fact, any other choice invalidates the commutativity of the diagram (1.6) since, in such a case, in the expression (3.4) the jump of some component of u would be multiplied by some power of ε. Therefore, letting ε go to zero, either the jump of such components disappears in the problem $(\mathcal{U}{\times}\mathcal{Z}, \mathsf{E}_{0,\kappa}, \mathsf{R}, \mathsf{q}_\kappa^0)$ or the product of the jumps by z would be constrained to be zero and we would end up with a mixed problem with a brittle delamination in some direction and an adhesive contact in the remaining.

The stored energy then writes as

$$\widetilde{\mathcal{E}}_{\varepsilon,\kappa}(t,u,z) = \begin{cases} \displaystyle\frac{1}{2}\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} \mathbb{C}e(u){:}e(u) - 2f^\varepsilon(t){\cdot}u\,\mathrm{d}x + \kappa\int_{\Gamma_{\rm C}^\varepsilon} zQ^\varepsilon\big([\![u]\!]_{\Gamma_{\rm C}^\varepsilon}\big)\,\mathrm{d}\mathscr{H}^2 & \text{if } (u,z) \in \mathcal{A}_\varepsilon^{\rm ad}(\psi^\varepsilon(t)), \\ +\infty & \text{else,} \end{cases} \tag{2.6}$$

where

$$\mathcal{A}^{\rm ad}_\varepsilon(\psi) = \big\{(u,z) \in W^{1,2}(\Omega^\varepsilon_1{\cup}\Omega^\varepsilon_2;\mathbb{R}^3) \times L^\infty(\Gamma^\varepsilon_{\rm C}) \ : \ \ u = \psi \text{ on } \Gamma^\varepsilon_{\rm Dir}, \\ 0 \le z \le 1, \ \ [\![u{\cdot}\nu]\!]_{\Gamma^\varepsilon_{\rm C}} \ge 0 \text{ on } \Gamma^\varepsilon_{\rm C}\big\}.$$

It should be noted that Definition 1.1 does not apply directly to the problems involving $\widetilde{\mathcal{E}}_\varepsilon$ and $\widetilde{\mathcal{E}}_{\varepsilon,\kappa}$ if the Dirichlet loading ψ^ε varies in time because the time derivative in (1.4b) is not well defined. To handle it, one must transform the problem to a time-independent Dirichlet loading problem. In this way, we consider a prolongation $w^\varepsilon(t)$ of $\psi^\varepsilon(t)$ to the whole Ω and then, instead of (2.1) and (2.6), we consider respectively

$$\mathcal{E}_\varepsilon(t,u,z) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\Omega^\varepsilon_1\cup\Omega^\varepsilon_2} \mathbb{C}e(u{+}w^\varepsilon(t)){:}e(u{+}w^\varepsilon(t)) - 2f^\varepsilon(t){\cdot}(u{+}w^\varepsilon(t))\,{\rm d}x & \text{if } (u,z) \in \mathcal{A}_\varepsilon, \\ +\infty & \text{else,} \end{cases} \tag{2.7}$$

where $\mathcal{A}_\varepsilon := \mathcal{A}_\varepsilon(0)$, and

$$\mathcal{E}_{\varepsilon,\kappa}(t,u,z) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\Omega^\varepsilon_1\cup\Omega^\varepsilon_2} \mathbb{C}e(u{+}w^\varepsilon(t)){:}e(u{+}w^\varepsilon(t)) - 2f^\varepsilon(t){\cdot}(u{+}w^\varepsilon(t))\,{\rm d}x & \\ \qquad\qquad +\kappa\displaystyle\int_{\Gamma^\varepsilon_{\rm C}} zQ^\varepsilon\big([\![u]\!]_{\Gamma^\varepsilon_{\rm C}}\big)\,{\rm d}\mathscr{H}^2 & \text{if } (u,z) \in \mathcal{A}^{\rm ad}_\varepsilon, \\ +\infty & \text{else,} \end{cases} \tag{2.8}$$

where $\mathcal{A}^{\rm ad}_\varepsilon := \mathcal{A}^{\rm ad}_\varepsilon(0)$. Then both models use the spaces $\mathcal{U}$ and $\mathcal{Z}$ in Definition 1.1 as

$$\mathcal{U}_\varepsilon = \big\{u \in W^{1,2}(\Omega^\varepsilon_1{\cup}\Omega^\varepsilon_2;\mathbb{R}^3) \ : \ \ u = 0 \text{ on } \Gamma^\varepsilon_{\rm Dir}\big\}, \\ \mathcal{Z}_\varepsilon = \big\{z \in L^\infty(\Gamma^\varepsilon_{\rm C}) \ : \ \ 0 \le z \le 1 \text{ on } \Gamma^\varepsilon_{\rm C}\big\};$$

here both $\mathcal{U}_\varepsilon$ and $\mathcal{Z}_\varepsilon$ are even subsets of Banach spaces and we will consider them equipped with standard norm, weak, and weak* topologies. Having an energetic solution $(u_\varepsilon, z_\varepsilon)$ to the problem $(\mathcal{U}_\varepsilon{\times}\mathcal{Z}_\varepsilon, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon, q^0)$ in accord to Definition 1.1, the shifted solution $(u_\varepsilon{+}w^\varepsilon, z_\varepsilon)$ will serve as an energetic solution to the original problem involving $\widetilde{\mathcal{E}}_\varepsilon$. Similar consideration concerns $\mathcal{E}_{\varepsilon,\kappa}$ vs. $\widetilde{\mathcal{E}}_{\varepsilon,\kappa}$. We will thus deal only with the transformed problems.

The following assertion has been proven in [35, Theorem 3.3].

Proposition 2.1 *Let $\varepsilon > 0$ be fixed, and let $q^0_\kappa = (u^0_\kappa, z^0_\kappa)$ be a sequence of initial conditions, with q^0_κ* stable *for all κ at time $t = 0$, that is*

$$\mathcal{E}_{\varepsilon,\kappa}(0, q^0_\kappa) < +\infty, \\ \mathcal{E}_{\varepsilon,\kappa}(0, q^0_\kappa) \le \mathcal{E}_{\varepsilon,\kappa}(0, \check{q}) + \mathcal{R}(\check{z}{-}z^0_\kappa) \quad \textit{for all } \check{q} = (\check{u}, \check{z}) \in \mathcal{Q}. \tag{2.9}$$

Assume moreover that $u^0_\kappa \rightharpoonup u^0$ in $W^{1,2}(\Omega^\varepsilon_1 \cup \Omega^\varepsilon_2;\mathbb{R}^3)$, $z^0_\kappa \overset{*}{\rightharpoonup} z^0$ in $L^\infty(\Gamma^\varepsilon_{\rm C})$ and $\mathcal{E}_{\varepsilon,\kappa}(0, q^0_\kappa) \to \mathcal{E}_\varepsilon(0, q^0)$, as $\kappa \to +\infty$. Then:

(i) the energetic solution $(u_{\varepsilon,\kappa}, z_{\varepsilon,\kappa})$ to the problem $(\mathcal{U}_\varepsilon{\times}\mathcal{Z}_\varepsilon, \mathcal{E}_{\varepsilon,\kappa}, \mathcal{R}_\varepsilon, q^0_\kappa)$ does exist;

(ii) also the energetic solution $(u_\varepsilon, z_\varepsilon)$ to the problem $(\mathcal{U}_\varepsilon{\times}\mathcal{Z}_\varepsilon, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon, q^0)$ does exist;

(iii) having a sequence of energetic solutions $\{(u_{\varepsilon,\kappa}, z_{\varepsilon,\kappa})\}_{\kappa>0}$ from (i), there exist a subsequence (not relabeled) and some $(u_\varepsilon, z_\varepsilon)$ such that

$$u_{\varepsilon,\kappa}(t) \rightharpoonup u_\varepsilon(t) \qquad \textit{in } W^{1,2}(\Omega^\varepsilon_1 \cup \Omega^\varepsilon_2;\mathbb{R}^3) \textit{ for all } t \in [0,T], \tag{2.10a}$$

$$z_{\varepsilon,\kappa}(t) \overset{*}{\rightharpoonup} z_\varepsilon(t) \qquad \textit{in } L^\infty(\Gamma^\varepsilon_{\rm C}) \textit{ for all } t \in [0,T], \tag{2.10b}$$

$$\partial_t\mathcal{E}_{\varepsilon,\kappa}(\cdot, u_{\varepsilon,\kappa}(\cdot), z_{\varepsilon,\kappa}(\cdot)) \rightharpoonup \partial_t\mathcal{E}_\varepsilon(\cdot, u_\varepsilon(\cdot), z_\varepsilon(\cdot)) \qquad \textit{in } L^1(0,T). \tag{2.10c}$$

Moreover, each $(u_\varepsilon, z_\varepsilon)$ obtained as a limit of such a selected subsequence is an energetic solution to the problem $(\mathcal{U}_\varepsilon{\times}\mathcal{Z}_\varepsilon, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon, q^0)$.

Remark 2.2 (Griffith concept.) A more conventional approach to cracks and brittle delamination is rather "geometrical", dealing with some delaminated area $\Gamma(t)$ of $\Gamma_{\rm C}^{\varepsilon}$ and a specific energy (in Joules per area) needed for increasing the delamination area. The dissipated energy (understood also as the so-called dissipation distance) is then

$$\mathcal{D}_\varepsilon(\Gamma_1,\Gamma_2) := \begin{cases} \displaystyle\int_{\Gamma_2\setminus\Gamma_1} a_\varepsilon(x)\mathrm{d}\mathscr{H}^2 & \text{if } \Gamma_1 \subset \Gamma_2 \subset \Gamma_{\rm C}^{\varepsilon}, \\ +\infty & \text{else,} \end{cases} \tag{2.11}$$

where $a_\varepsilon \in L^\infty(\Gamma_{\rm C}^{\varepsilon})$ is from (2.2). Such a geometrical setting was used for large strains in [19] for polyconvex materials and in [8] for quasiconvex materials, and in the small-strain setting also in e.g. [17, 33, 37, 38]. The philosophy of quasistatic evolution is related with the *Griffith criterion* [9] saying that the crack grows as soon as the energy release is bigger than the fracture toughness, here determined by a_ε in (2.11). The relation between the "geometrical" concept used in (2.11) and the previous "functional" concept is that, if z takes values only 0 or 1, i.e. always $z = \chi_\Gamma$ for some $\Gamma \subset \Gamma_{\rm C}^{\varepsilon}$, then

$$\mathcal{D}_\varepsilon(\Gamma_1,\Gamma_2) = \mathcal{R}_\varepsilon(z_2 - z_1) \quad \text{with } z_i = \chi_{\Gamma_i},\ i = 1,2. \tag{2.12}$$

It has been proven in [29] that any energetic solution $(u_\varepsilon, z_\varepsilon)$ to the brittle delamination problem $(\mathcal{U}_\varepsilon \times \mathcal{Z}_\varepsilon, \mathcal{E}_\varepsilon, \mathcal{R}_\varepsilon, q^0)$ is indeed of Griffith's type in the sense that z_ε takes only the values 0 and 1. Let us remark that the crack path is here assumed to be a-priori prescribed; as in the above mentioned references. Terms like "delamination" or "debonding" are often used to highlight that the path is given, so to distinguish these models from those with a free crack path whose mathematical formulation is completely different, cf. [6, 7, 11, 22].

3 The rescaled problems

To perform the dimension reduction it is convenient to make a change of variables in order to work on domains independent of ε. Let $\Omega := \Omega^1$, $\Omega_1 := \Omega_1^1$, $\Omega_2 := \Omega_2^1$, $\Gamma_{\rm C} := \Gamma_{\rm C}^1$, and $\Gamma_{\rm Dir} := \Gamma_{\rm Dir}^1$.

For any $\varepsilon > 0$, let $p_\varepsilon : \overline{\Omega_1 \cup \Omega_2} \to \overline{\Omega_1^\varepsilon \cup \Omega_2^\varepsilon}$ be defined by

$$p_\varepsilon(x_1, x_2, x_3) := (x_1, x_2, \varepsilon x_3).$$

The variables on the fixed domain, $\Omega_1 \cup \Omega_2$, will be denoted by using the Sans font. Thus for $u \in W^{1,2}(\Omega_1^\varepsilon \cup \Omega_2^\varepsilon; \mathbb{R}^3)$ we let $\mathsf{u} \in W^{1,2}(\Omega_1 \cup \Omega_2; \mathbb{R}^3)$ be defined by $\mathsf{u}_\alpha := \frac{1}{\varepsilon} u_\alpha \circ p_\varepsilon$, $\alpha = 1, 2$, and $\mathsf{u}_3 := u_3 \circ p_\varepsilon$, while for $z \in L^\infty(\Gamma_{\rm C}^\varepsilon)$ we set $\mathsf{z} := z \circ p_\varepsilon \in L^\infty(\Gamma_{\rm C})$. With this notation we have

$$\frac{1}{\varepsilon} e(u) \circ p_\varepsilon = \begin{pmatrix} e(\mathsf{u})_{\alpha\beta} & \frac{1}{\varepsilon} e(\mathsf{u})_{\alpha 3} \\ \frac{1}{\varepsilon} e(\mathsf{u})_{3\beta} & \frac{1}{\varepsilon^2} e(\mathsf{u})_{33} \end{pmatrix} =: e^\varepsilon(\mathsf{u}). \tag{3.1}$$

In order to keep the displacements bounded we need to rescale the forces and the boundary conditions with ε. We assume that there exists an $\mathsf{f} \in C^1([0,T]; L^2(\Omega; \mathbb{R}^3))$ such that

$$\varepsilon \mathsf{f}_\alpha(t) = f_\alpha^\varepsilon(t) \circ p_\varepsilon \quad \text{ and } \quad \varepsilon^2 \mathsf{f}_3(t) = f_3^\varepsilon(t) \circ p_\varepsilon.$$

Concerning the boundary conditions we need to require that $e^\varepsilon(\mathsf{w}^\varepsilon)(t)$ is bounded in $L^2(\Omega_1 \cup \Omega_2; \mathbb{R}^{3\times 3})$, where we have set $\mathsf{w}_\alpha^\varepsilon := \frac{1}{\varepsilon} w_\alpha^\varepsilon \circ p_\varepsilon$ and $\mathsf{w}_3^\varepsilon := w_3^\varepsilon \circ p_\varepsilon$. The simplest way to fulfill this requirement is to pose $\mathsf{w}^\varepsilon = \mathsf{w}$, with w satisfying the following Kirchhoff-Love assumption which is fundamental in plate theory, cf. [5]:

$$\begin{aligned} \mathsf{w}_\alpha(t, x_1, x_2, x_3) &:= \eta_\alpha(t, x_1, x_2) - x_3 \tfrac{\partial \zeta}{\partial x_\alpha}(t, x_1, x_2), \\ \mathsf{w}_3(t, x_1, x_2, x_3) &:= \zeta(t, x_1, x_2), \end{aligned} \tag{3.2}$$

where $\zeta \in C^1([0,T]; W^{2,2}(\Omega; \mathbb{R}^3))$ and $\eta_\alpha \in C^1([0,T]; W^{1,2}(\Omega; \mathbb{R}^3))$.

Let us remark that with this choice we have

$$e^\varepsilon(\mathsf{w}^\varepsilon)_{i3} = 0, \quad i = 1,2,3, \tag{3.3}$$

hence, in particular, $e^\varepsilon(\mathsf{w}^\varepsilon) = e^\varepsilon(\mathsf{w}) = e(\mathsf{w})$ for any ε.

For $(u,z) \in \mathcal{A}^{\mathrm{ad}}_\varepsilon$ we have

$$\begin{aligned}\mathcal{E}_{\varepsilon,\kappa}(t,u,z) &= \frac{1}{2}\int_{\Omega_1^\varepsilon\cup\Omega_2^\varepsilon} \mathbb{C}e(u{+}w^\varepsilon(t)){:}e(u{+}w^\varepsilon(t)) - 2f^\varepsilon(t)\cdot(u{+}w^\varepsilon(t))\,\mathrm{d}x + \kappa\int_{\Gamma_{\mathrm{C}}^\varepsilon} zQ^\varepsilon\big(\llbracket u\rrbracket_{\Gamma_{\mathrm{C}}^\varepsilon}\big)\,\mathrm{d}\mathscr{H}^2\\ &= \frac{\varepsilon^3}{2}\int_{\Omega_1\cup\Omega_2} \mathbb{C}e^\varepsilon(\mathsf{u}{+}\mathsf{w}(t)){:}e^\varepsilon(\mathsf{u}{+}\mathsf{w}(t)) - 2\mathsf{f}(t)\cdot(\mathsf{u}{+}\mathsf{w}(t))\,\mathrm{d}x + \kappa\varepsilon^3\int_{\Gamma_{\mathrm{C}}} \mathsf{z}\big|\llbracket \mathsf{u}\rrbracket_{\Gamma_{\mathrm{C}}}\big|^2\,\mathrm{d}\mathscr{H}^2 =: \varepsilon^3\mathsf{E}_{\varepsilon,\kappa}(t,\mathsf{u},\mathsf{z}).\end{aligned}$$

We have therefore

$$\mathsf{E}_{\varepsilon,\kappa}(t,\mathsf{u},\mathsf{z}) = \begin{cases}\displaystyle\frac{1}{2}\int_{\Omega_1\cup\Omega_2}\mathbb{C}e^\varepsilon(\mathsf{u}{+}\mathsf{w}(t)){:}e^\varepsilon(\mathsf{u}{+}\mathsf{w}(t)) - 2\mathsf{f}(t)\cdot(\mathsf{u}{+}\mathsf{w}(t))\,\mathrm{d}x + \kappa\int_{\Gamma_{\mathrm{C}}}\mathsf{z}\big|\llbracket\mathsf{u}\rrbracket_{\Gamma_{\mathrm{C}}}\big|^2\,\mathrm{d}\mathscr{H}^2 & \text{if } (\mathsf{u},\mathsf{z})\in\mathsf{A}^{\mathrm{ad}},\\ +\infty & \text{else,}\end{cases} \tag{3.4}$$

where

$$\mathsf{A}^{\mathrm{ad}} = \big\{(\mathsf{u},\mathsf{z})\in W^{1,2}(\Omega_1{\cup}\Omega_2;\mathbb{R}^3)\times L^\infty(\Gamma_{\mathrm{C}}):\ \mathsf{u}=0 \text{ on } \Gamma_{\mathrm{Dir}},\ \ 0\le\mathsf{z}\le1,\ \ \llbracket\mathsf{u}\cdot\nu\rrbracket_{\Gamma_{\mathrm{C}}}\ge0 \text{ on } \Gamma_{\mathrm{C}}\big\}.$$

If, instead, $(u,z)\in\mathcal{A}_\varepsilon$ then we have

$$\mathcal{E}_\varepsilon(t,u,z) = \varepsilon^3\mathsf{E}_\varepsilon(t,\mathsf{u},\mathsf{z}) \tag{3.5}$$

where

$$\mathsf{E}_\varepsilon(t,\mathsf{u},\mathsf{z}) := \begin{cases}\displaystyle\frac{1}{2}\int_{\Omega_1\cup\Omega_2}\mathbb{C}e^\varepsilon(\mathsf{u}{+}\mathsf{w}(t)){:}e^\varepsilon(\mathsf{u}{+}\mathsf{w}(t)) - 2\mathsf{f}(t)\cdot(\mathsf{u}{+}\mathsf{w}(t))\,\mathrm{d}x & \text{if } (\mathsf{u},\mathsf{z})\in\mathsf{A},\\ +\infty & \text{else,}\end{cases} \tag{3.6}$$

and

$$\mathsf{A} = \big\{(\mathsf{u},\mathsf{z})\in W^{1,2}(\Omega_1{\cup}\Omega_2;\mathbb{R}^3)\times L^\infty(\Gamma_{\mathrm{C}})\ :\ \mathsf{u}=0 \text{ on } \Gamma_{\mathrm{Dir}},\ \ 0\le\mathsf{z}\le1,\ \ \llbracket\mathsf{u}\cdot\nu\rrbracket_{\Gamma_{\mathrm{C}}}\ge0,\ \ \mathsf{z}\llbracket u\rrbracket_{\Gamma_{\mathrm{C}}}=0 \text{ on } \Gamma_{\mathrm{C}}\big\}.$$

Similarly we rescale the dissipation energy; we assume that there exists a measurable function $\mathsf{a}\in L^1(\Gamma_{\mathrm{C}})$ such that

$$\frac{a_\varepsilon\circ p_\varepsilon}{\varepsilon^2} = \mathsf{a} \tag{3.7}$$

for any ε, so that

$$\mathcal{R}_\varepsilon(\dot z) = \varepsilon^3\mathsf{R}(\dot{\mathsf{z}}) \tag{3.8}$$

where

$$\mathsf{R}(\dot{\mathsf{z}}) := \begin{cases}\displaystyle\int_{\Gamma_{\mathrm{C}}}\mathsf{a}|\dot{\mathsf{z}}|\,\mathrm{d}\mathscr{H}^2 & \text{if } \dot{\mathsf{z}}\le0 \text{ on } \Gamma_{\mathrm{C}},\\ +\infty & \text{else.}\end{cases} \tag{3.9}$$

In fact, there is an infinite number of rescalings of the dissipation energy. We have chosen the one, namely (3.7), that yields the same scaling of the stored energy, i.e. ε^3, cf. (3.5) and (3.8) and thus we obtain a non-trivial dissipation in the limit problem. We also assume that $a_{\varepsilon,\min} = \varepsilon^2\mathsf{a}_{\min}$ for a suitable constant $\mathsf{a}_{\min}>0$ so that $\mathsf{a}(x)\ge\mathsf{a}_{\min}$ for $\mathscr{H}^2$-a.e. $x\in\Gamma_{\mathrm{C}}$.

4 Some semicontinuity properties

In this section we state and prove a couple of lemmas concerning upper and lower bounds for the limit of the energies as ε goes to 0 which shall be useful in identifying the limit problem.

To state our results it is useful to introduce the following tensor components

$$\mathbb{C}^0_{\alpha\beta\gamma\delta} := \mathbb{C}_{\alpha\beta\gamma\delta} - \frac{\mathbb{C}_{\alpha\beta 33}\mathbb{C}_{\gamma\delta 33}}{\mathbb{C}_{3333}}, \qquad \alpha,\beta,\gamma,\delta = 1,2, \tag{4.1}$$

and remark that, setting

$$g(e) := \frac{1}{2}\mathbb{C}e{:}e \qquad \text{for every } e \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}$$

under assumptions (2.4) and (2.5) we have (see also [1])

$$\frac{1}{2}\mathbb{C}^0\tilde{e}{:}\tilde{e} = \min_{\tilde{\eta}\in\mathbb{R}^2,\eta_3\in\mathbb{R}} g\begin{pmatrix} \tilde{e} & \tilde{\eta} \\ \tilde{\eta}^\top & \eta_3 \end{pmatrix} \qquad \text{for every } \tilde{e} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \tag{4.2}$$

where the minimum is achieved for $\tilde{\eta} = 0$ and

$$\eta_3 = -\sum_{\alpha,\beta=1}^{2} \frac{\mathbb{C}_{33\alpha\beta}\tilde{e}_{\alpha\beta}}{\mathbb{C}_{3333}}.$$

Moreover, let us denote by $\tilde{e}(\mathsf{u})$ the $2{\times}2$-matrix with components

$$\tilde{e}(\mathsf{u})_{\alpha\beta} := e(\mathsf{u})_{\alpha\beta}.$$

Finally, let us introduce the following space of Kirchhoff-Love displacements

$$W^{1,2}_{\mathrm{KL}}(\Omega_1{\cup}\Omega_2;\mathbb{R}^3) := \big\{\mathsf{u} \in W^{1,2}(\Omega_1{\cup}\Omega_2;\mathbb{R}^3) : \ e(\mathsf{u})_{i3} = 0 \ \text{ for } i = 1,2,3\big\} \tag{4.3}$$

which contains, in fact, the effective domain of the following limiting energy functionals

$$\mathsf{E}_0(t,\mathsf{u},\mathsf{z}) := \begin{cases} \dfrac{1}{2}\displaystyle\int_{\Omega_1\cup\Omega_2} \mathbb{C}^0\tilde{e}(\mathsf{u}{+}\mathsf{w}(t)){:}\tilde{e}(\mathsf{u}{+}\mathsf{w}(t)) - 2\mathsf{f}(t){\cdot}(\mathsf{u}{+}\mathsf{w}(t))\,\mathrm{d}x & \text{if } (\mathsf{u},\mathsf{z}) \in \mathsf{A}_{\mathrm{KL}}, \\ +\infty & \text{else,} \end{cases} \tag{4.4}$$

where

$$\mathsf{A}_{\mathrm{KL}} = \big\{(\mathsf{u},\mathsf{z}) \in \mathsf{A} \ : \ \mathsf{u} \in W^{1,2}_{\mathrm{KL}}(\Omega_1{\cup}\Omega_2;\mathbb{R}^3)\big\},$$

and

$$\mathsf{E}_{0,\kappa}(t,\mathsf{u},\mathsf{z}) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\Omega_1\cup\Omega_2} \mathbb{C}^0\tilde{e}(\mathsf{u}{+}\mathsf{w}(t)){:}\tilde{e}(\mathsf{u}{+}\mathsf{w}(t)) - 2\mathsf{f}(t){\cdot}(\mathsf{u}{+}\mathsf{w}(t))\,\mathrm{d}x + \displaystyle\int_{\Gamma_{\mathrm{C}}} \mathsf{z}\kappa\big|[\![\mathsf{u}]\!]_{\Gamma_{\mathrm{C}}}\big|^2 \mathrm{d}\mathscr{H}^2 & \text{if } (\mathsf{u},\mathsf{z}) \in \mathsf{A}^{\mathrm{ad}}_{\mathrm{KL}}, \\ +\infty & \text{else,} \end{cases} \tag{4.5}$$

where

$$\mathsf{A}^{\mathrm{ad}}_{\mathrm{KL}} = \big\{(\mathsf{u},\mathsf{z}) \in \mathsf{A}^{\mathrm{ad}} \ : \ \mathsf{u} \in W^{1,2}_{\mathrm{KL}}(\Omega_1{\cup}\Omega_2;\mathbb{R}^3)\big\}.$$

The evolution problems will be considered on the spaces

$$\mathcal{U} = \big\{\mathsf{u} \in W^{1,2}(\Omega_1{\cup}\Omega_2;\mathbb{R}^3) \ : \ \mathsf{u} = 0 \text{ on } \Gamma_{\mathrm{Dir}}\big\},$$
$$\mathcal{Z} = \big\{\mathsf{z} \in L^\infty(\Gamma_{\mathrm{C}}) \ : \ 0 \le \mathsf{z} \le 1 \text{ a.e.}\big\}.$$

The next lemma states that the sets A and A^{ad} are closed with respect to the weak convergence, and that the work done by loads and the interfacial energy are continuous with respect to the convergence in time and the weak convergences in (u,z).

Lemma 4.1 *The spaces $\mathcal{U}$ and $\mathcal{Z}$ are closed with respect to the weak and weak* convergence, respectively. Moreover, if $t_\varepsilon \to t$, $\mathsf{z}_\varepsilon \overset{*}{\rightharpoonup} \mathsf{z}$ in $L^\infty(\Gamma_{\mathrm{C}})$, $0 \le \mathsf{z}_\varepsilon \le 1$ a.e., and $\mathsf{u}_\varepsilon \rightharpoonup \mathsf{u}$ in $W^{1,2}(\Omega_1 \cup \Omega_2; \mathbb{R}^3)$ as $\varepsilon \to 0^+$, then*

$$\llbracket \mathsf{u}_\varepsilon \cdot \nu \rrbracket_{\Gamma_{\mathrm{C}}} \ge 0 \quad \text{on } \Gamma_{\mathrm{C}} \quad \Longrightarrow \quad \llbracket \mathsf{u} \cdot \nu \rrbracket_{\Gamma_{\mathrm{C}}} \ge 0 \quad \text{on } \Gamma_{\mathrm{C}}, \tag{4.6a}$$

$$\mathsf{z}_\varepsilon \llbracket \mathsf{u}_\varepsilon \rrbracket_{\Gamma_{\mathrm{C}}} = 0 \quad \text{on } \Gamma_{\mathrm{C}} \quad \Longrightarrow \quad \mathsf{z} \llbracket \mathsf{u} \rrbracket_{\Gamma_{\mathrm{C}}} = 0 \quad \text{on } \Gamma_{\mathrm{C}}, \tag{4.6b}$$

$$\int_{\Gamma_{\mathrm{C}}} \mathsf{z}_\varepsilon \kappa \big| \llbracket \mathsf{u}_\varepsilon \rrbracket_{\Gamma_{\mathrm{C}}} \big|^2 \, \mathrm{d}\mathscr{H}^2 \to \int_{\Gamma_{\mathrm{C}}} \mathsf{z} \kappa \big| \llbracket \mathsf{u} \rrbracket_{\Gamma_{\mathrm{C}}} \big|^2 \, \mathrm{d}\mathscr{H}^2, \tag{4.6c}$$

$$\int_{\Omega_1 \cup \Omega_2} \mathsf{f}(t_\varepsilon) \cdot (\mathsf{u}_\varepsilon + \mathsf{w}(t_\varepsilon)) \, \mathrm{d}x \to \int_{\Omega_1 \cup \Omega_2} \mathsf{f}(t) \cdot (\mathsf{u} + \mathsf{w}(t)) \, \mathrm{d}x. \tag{4.6d}$$

Proof: The first assertion follows from the continuity of the trace and the basic properties of the weak* convergence in $L^\infty(\Gamma_{\mathrm{C}})$.

The convergence assumption on the sequence (u^ε) implies that the traces of u^ε on Γ_{C} converge strongly in L^2 and this implies (4.6a-c), while (4.6d) follows by the continuity of f and w with respect to t which implies $\mathsf{f}(t_\varepsilon) \to \mathsf{f}(t)$ and $\mathsf{w}(t_\varepsilon) \to \mathsf{w}(t)$ in $L^2(\Omega; \mathbb{R}^3)$.

In the next lemma we prove a lower bound for the limit energies. A similar result was also proven in [3].

Lemma 4.2 (Lower semicontinuity.) *If $t_\varepsilon \to t$, $\mathsf{z}_\varepsilon \overset{*}{\rightharpoonup} \mathsf{z}$ in $L^\infty(\Gamma_{\mathrm{C}})$, $0 \le \mathsf{z}_\varepsilon \le 1$ a.e., and $\mathsf{u}_\varepsilon \rightharpoonup \mathsf{u}$ in $W^{1,2}(\Omega_1 \cup \Omega_2; \mathbb{R}^3)$ as $\varepsilon \to 0^+$, then*

$$\liminf_{\varepsilon \to 0^+} \mathsf{E}_\varepsilon(t_\varepsilon, \mathsf{u}_\varepsilon, \mathsf{z}_\varepsilon) \ge \mathsf{E}_0(t, \mathsf{u}, \mathsf{z}) \tag{4.7}$$

and, for any $\kappa > 0$,

$$\liminf_{\varepsilon \to 0^+} \mathsf{E}_{\varepsilon,\kappa}(t_\varepsilon, \mathsf{u}_\varepsilon, \mathsf{z}_\varepsilon) \ge \mathsf{E}_{0,\kappa}(t, \mathsf{u}, \mathsf{z}). \tag{4.8}$$

Proof: Under the assumption that the liminf on the left-hand sides be finite, by the positive definiteness of $\mathbb{C}$ (see (2.3)) and the continuity of w with respect to t, it follows that

$$\sup_\varepsilon \big\| e^\varepsilon(\mathsf{u}_\varepsilon) \big\|_{L^2} < +\infty$$

and hence

$$\big\| e(\mathsf{u}_\varepsilon)_{i3} \big\|_{L^2} \le C\varepsilon, \quad i = 1, 2, 3.$$

Thus, passing to the limit, we obtain

$$e(\mathsf{u})_{i3} = 0, \quad i = 1, 2, 3.$$

which implies $\mathsf{u} \in W^{1,2}_{\mathrm{KL}}(\Omega_1 \cup \Omega_2; \mathbb{R}^3)$. By Lemma 4.1 we have that also the right hand side in (4.7) and (4.8) are finite and that it suffices to prove that

$$\liminf_{\varepsilon \to 0^+} \int_{\Omega_1 \cup \Omega_2} \mathbb{C} e^\varepsilon(\mathsf{u}_\varepsilon + \mathsf{w}(t_\varepsilon)) : e^\varepsilon(\mathsf{u}_\varepsilon + \mathsf{w}(t_\varepsilon)) \, \mathrm{d}x \ge \int_{\Omega_1 \cup \Omega_2} \mathbb{C}^0 \tilde{e}(\mathsf{u} + \mathsf{w}(t)) : \tilde{e}(\mathsf{u} + \mathsf{w}(t)) \, \mathrm{d}x.$$

In fact, noticing that, for $\alpha, \beta = 1, 2$,

$$e^\varepsilon(\mathsf{u}_\varepsilon)_{\alpha\beta} = e(\mathsf{u}_\varepsilon)_{\alpha\beta} \rightharpoonup e(\mathsf{u})_{\alpha\beta} \quad \text{in } L^2(\Omega_1 \cup \Omega_2),$$

using property (4.2) and the continuity of w with respect to t we find

$$\begin{aligned} \liminf_{\varepsilon \to 0^+} \int_{\Omega_1 \cup \Omega_2} \mathbb{C} e^\varepsilon(\mathsf{u}_\varepsilon + \mathsf{w}(t_\varepsilon)) : e^\varepsilon(\mathsf{u}_\varepsilon + \mathsf{w}(t_\varepsilon)) \, \mathrm{d}x &\ge \liminf_{\varepsilon \to 0^+} \int_{\Omega_1 \cup \Omega_2} \mathbb{C}^0 \tilde{e}(\mathsf{u}_\varepsilon + \mathsf{w}(t_\varepsilon)) : \tilde{e}(\mathsf{u}_\varepsilon + \mathsf{w}(t_\varepsilon)) \, \mathrm{d}x \\ &\ge \int_{\Omega_1 \cup \Omega_2} \mathbb{C}^0 \tilde{e}(\mathsf{u} + \mathsf{w}(t)) : \tilde{e}(\mathsf{u} + \mathsf{w}(t)) \, \mathrm{d}x, \end{aligned} \tag{4.9}$$

which concludes the proof.

The next lemma is fundamental to apply the Γ-convergence scheme developed in [30].

Lemma 4.3 (Mutual recovery sequence.) *Let (ε_n) be a sequence such that $0 < \varepsilon_n \to 0$. Let $t \in [0,T]$, $\mathsf{z} \in L^\infty(\Gamma_{\rm C})$ and $0 \le \mathsf{z} \le 1$ a.e.. For every $(t_n, \mathsf{z}_n) \to (t, \mathsf{z})$, $\mathsf{z}_n \in L^\infty(\Gamma_{\rm C})$, $0 \le \mathsf{z}_n \le 1$ a.e., every $\check{\mathsf{z}} \in L^\infty(\Gamma_{\rm C})$ and every $\check{\mathsf{u}} \in W^{1,2}(\Omega_1 \cup \Omega_2; \mathbb{R}^3)$, there exist $\check{\mathsf{z}}_n \in \mathcal{Z}$ and $\check{\mathsf{u}}_n \in \mathcal{U}$ such that $\check{\mathsf{u}}_n \rightharpoonup \check{\mathsf{u}}$ in $W^{1,2}(\Omega_1 \cup \Omega_2; \mathbb{R}^3)$, $\check{\mathsf{z}}_n \overset{*}{\rightharpoonup} \check{\mathsf{z}}$ in $L^\infty(\Gamma_{\rm C})$ and*

$$\limsup_{n\to+\infty} \big[\mathsf{E}_{\varepsilon_n}(t_n, \check{\mathsf{u}}_n, \check{\mathsf{z}}_n) + \mathsf{R}(\check{\mathsf{z}}_n - \mathsf{z}_n)\big] \le \mathsf{E}_0(t, \check{\mathsf{u}}, \check{\mathsf{z}}) + \mathsf{R}(\check{\mathsf{z}} - \mathsf{z}) \tag{4.10}$$

and, for any $\kappa > 0$,

$$\limsup_{n\to+\infty} \big[\mathsf{E}_{\varepsilon_n,\kappa}(t_n, \check{\mathsf{u}}_n, \check{\mathsf{z}}_n) + \mathsf{R}(\check{\mathsf{z}}_n - \mathsf{z}_n)\big] \le \mathsf{E}_{0,\kappa}(t, \check{\mathsf{u}}, \check{\mathsf{z}}) + \mathsf{R}(\check{\mathsf{z}} - \mathsf{z}). \tag{4.11}$$

Let us remark that the mutual recovery sequence condition stated in [30] is

$$\limsup_{n\to+\infty} \big[\mathsf{E}_{\varepsilon_n,\kappa}(t_n, \check{\mathsf{u}}_n, \check{\mathsf{z}}_n) + \mathsf{R}(\check{\mathsf{z}}_n - \mathsf{z}_n) - \mathsf{E}_{\varepsilon_n,\kappa}(t_n, \mathsf{u}_n, \mathsf{z}_n)\big] \le \mathsf{E}_{0,\kappa}(t, \check{\mathsf{u}}, \check{\mathsf{z}}) + \mathsf{R}(\check{\mathsf{z}} - \mathsf{z}) - \mathsf{E}_{0,\kappa}(t, \mathsf{u}, \mathsf{z})$$

where $(\mathsf{u}_n, \mathsf{z}_n)$ is a stable sequence converging to (u, z), and which is slightly weaker than (4.11), as proven in [30, Prop.2.2]. An analogous remark applies to (4.10) combined with (4.7). In spite of this, it is simpler for us to prove the stronger conditions (4.10) and (4.11) since we can provide convergence of the dissipation term.

Proof: [P r o o f of Lemma 4.3] First of all we notice that inequalities (4.10) and (4.11) are nontrivial only when the right-hand sides are finite. In particular this implies that $\check{\mathsf{u}} \in \mathcal{U}$, $\check{\mathsf{z}} \in \mathcal{Z}$ and $\mathsf{z} \ge \check{\mathsf{z}} \ge 0$ on $\Gamma_{\rm C}$.

Inspired by [35], let us define

$$\check{\mathsf{z}}_n := \begin{cases} \mathsf{z}_n \check{\mathsf{z}}/\mathsf{z} & \text{if } \mathsf{z} > 0, \\ 0 & \text{if } \mathsf{z} = 0. \end{cases} \tag{4.12}$$

Then we have $0 \le \check{\mathsf{z}}_n \le 1$ a.e. and $\check{\mathsf{z}}_n \overset{*}{\rightharpoonup} \check{\mathsf{z}}$ in $L^\infty(\Gamma_{\rm C})$. Moreover, since $\check{\mathsf{z}}_n - \mathsf{z}_n \le 0$, then

$$\lim_{n\to+\infty} \mathsf{R}(\check{\mathsf{z}}_n - \mathsf{z}_n) = \lim_{n\to+\infty} \int_{\Gamma_{\rm C}} \mathsf{a}(\mathsf{z}_n - \check{\mathsf{z}}_n)\,{\rm d}\mathscr{H}^2 = \int_{\Gamma_{\rm C}} \mathsf{a}(\mathsf{z} - \check{\mathsf{z}})\,{\rm d}\mathscr{H}^2 = \mathsf{R}(\check{\mathsf{z}} - \mathsf{z}).$$

Thus, to prove the claim it suffices to deal with the convergence of the energies, that is, to prove that

$$\limsup_{n\to+\infty} \mathsf{E}_{\varepsilon_n}(t_n, \check{\mathsf{u}}_n, \check{\mathsf{z}}_n) \le \mathsf{E}_0(t, \check{\mathsf{u}}, \check{\mathsf{z}}), \qquad \limsup_{n\to+\infty} \mathsf{E}_{\varepsilon_n,\kappa}(t_n, \check{\mathsf{u}}_n, \check{\mathsf{z}}_n) \le \mathsf{E}_{0,\kappa}(t, \check{\mathsf{u}}, \check{\mathsf{z}}). \tag{4.13}$$

Inspired by [3], for fixed t we let

$$\psi := -\sum_{\alpha,\beta=1}^{2} \frac{\mathbb{C}_{33\alpha\beta}\tilde{e}(\check{\mathsf{u}}+\mathsf{w}(t))_{\alpha\beta}}{\mathbb{C}_{3333}},$$

and choose $\psi_n \in C_0^\infty(\Omega)$ such that $\psi_n \to \psi$ and $\varepsilon_n \partial\psi_n(t,\cdot)/\partial x_\alpha \to 0$ in $L^2(\Omega)$. Set

$$\eta_n(x_1, x_2, x_3) := \int_0^{x_3} \psi_n(x_1, x_2, s)\,{\rm d}s$$

and

$$\left.\begin{aligned} (\check{\mathsf{u}}_n)_\alpha &:= \check{\mathsf{u}}_\alpha, \qquad \alpha = 1, 2, \\ (\check{\mathsf{u}}_n)_3 &:= \check{\mathsf{u}}_3 + \varepsilon_n^2 \eta_n. \end{aligned}\right\} \tag{4.14}$$

In this way we have $\check{\mathsf{u}}_n \in W^{1,2}(\Omega_1 \cup \Omega_2; \mathbb{R}^3)$ and $\check{\mathsf{u}}_n \rightharpoonup \check{\mathsf{u}}$ in $W^{1,2}(\Omega_1 \cup \Omega_2; \mathbb{R}^3)$ (actually this convergence is strong).

If $\mathsf{E}_0(t, \check{\mathsf{u}}, \check{\mathsf{z}}) < +\infty$ or $\mathsf{E}_{0,\kappa}(t, \check{\mathsf{u}}, \check{\mathsf{z}}) < +\infty$ then $\mathsf{z}_n \in L^\infty(\Gamma_{\rm C})$, $0 \le \mathsf{z}_n \le 1$ a.e., and

$$\begin{aligned} &[\![\check{\mathsf{u}}_n \cdot \nu]\!]_{\Gamma_{\rm C}} = [\![\check{\mathsf{u}} \cdot \nu]\!]_{\Gamma_{\rm C}} \ge 0, \\ &\check{\mathsf{u}}_n = \check{\mathsf{u}} = 0 \text{ on } \Gamma_{\rm Dir}. \end{aligned}$$

If $\mathsf{E}_0(t,\check{\mathsf{u}},\check{\mathsf{z}})<+\infty$ then moreover

$$\mathsf{z}_n[\![\check{\mathsf{u}}_n]\!]_{\Gamma_\mathrm{C}}=\mathsf{z}[\![\check{\mathsf{u}}]\!]_{\Gamma_\mathrm{C}}=0 \text{ on } \Gamma_\mathrm{C}.$$

Thus, if $\mathsf{E}_0(t,\check{\mathsf{u}},\check{\mathsf{z}})<+\infty$ or $\mathsf{E}_{0,\kappa}(t,\check{\mathsf{u}},\check{\mathsf{z}})<+\infty$, respectively, then

$$\mathsf{E}_{\varepsilon_n}(t_n,\check{\mathsf{u}}_n,\check{\mathsf{z}}_n)=\frac{1}{2}\int_{\Omega_1\cup\Omega_2}\mathbb{C}e^{\varepsilon_n}(\check{\mathsf{u}}_n{+}\mathsf{w}(t_n)){:}e^{\varepsilon_n}(\check{\mathsf{u}}_n{+}\mathsf{w}(t_n))-2\mathsf{f}(t_n){\cdot}(\check{\mathsf{u}}_n{+}\mathsf{w}(t_n))\,\mathrm{d}x,$$

while

$$\mathsf{E}_{\varepsilon_n,\kappa}(t_n,\check{\mathsf{u}}_n,\check{\mathsf{z}}_n)=\frac{1}{2}\int_{\Omega_1\cup\Omega_2}\mathbb{C}e^{\varepsilon_n}(\check{\mathsf{u}}_n{+}\mathsf{w}(t_n)){:}e^{\varepsilon_n}(\check{\mathsf{u}}_n{+}\mathsf{w}(t_n))-2\mathsf{f}(t_n){\cdot}(\check{\mathsf{u}}_n{+}\mathsf{w}(t_n))\,\mathrm{d}x+\kappa\int_{\Gamma_\mathrm{C}}\mathsf{z}_n\big|[\![\check{\mathsf{u}}_n]\!]_{\Gamma_\mathrm{C}}\big|^2\,\mathrm{d}\mathscr{H}^2.$$

By (3.1) and since $\check{\mathsf{u}}$ and w are in $W^{1,2}_{\mathrm{KL}}(\Omega_1{\cup}\Omega_2;\mathbb{R}^3)$, we have

$$e^{\varepsilon_n}(\check{\mathsf{u}}_n{+}\mathsf{w}(t_n))=\begin{pmatrix} e(\check{\mathsf{u}}{+}\mathsf{w}(t_n))_{\alpha\beta} & \dfrac{\varepsilon_n}{2}\dfrac{\partial\eta_n}{\partial x_\alpha}\\ \mathrm{sym} & \psi_n\end{pmatrix}.$$

Taking the limit as $n\to+\infty$ we have

$$\lim_{n\to+\infty}\left\|e^{\varepsilon_n}\big(\check{\mathsf{u}}_n{+}\mathsf{w}(t_n)\big)-\begin{pmatrix}\tilde{e}(\check{\mathsf{u}}{+}\mathsf{w}(t)) & 0\\ 0 & -\sum_{\alpha,\beta=1}^2\mathbb{C}_{33\alpha\beta}\tilde{e}(\check{\mathsf{u}}{+}\mathsf{w}(t))_{\alpha\beta}/\mathbb{C}_{3333}\end{pmatrix}\right\|_{L^2}=0,$$

and therefore, concerning the bulk part, we have

$$\lim_{n\to+\infty}\int_{\Omega_1\cup\Omega_2}\mathbb{C}e^{\varepsilon_n}(\check{\mathsf{u}}_n{+}\mathsf{w}(t_n)){:}e^{\varepsilon_n}(\check{\mathsf{u}}_n{+}\mathsf{w}(t_n))\,\mathrm{d}x=\int_{\Omega_1\cup\Omega_2}\mathbb{C}^0\tilde{e}(\check{\mathsf{u}}{+}\mathsf{w}(t)){:}\tilde{e}(\check{\mathsf{u}}{+}\mathsf{w}(t))\,\mathrm{d}x.$$

The inequalities (4.13) follow then by applying Lemma 4.1, and the proof is concluded.

5 Convergence of solutions

The aim of this section is to show, assuming $\mathsf{q}^0_\kappa\rightharpoonup\mathsf{q}^0$ in $\mathcal{Q}$, the convergences depicted in the diagramme (1.6). The notation in use is that of [30]. According to what we have done above, we denote by $\mathcal{Q}$ the topological product of the spaces $\mathcal{U}$ and $\mathcal{Z}$ endowed, respectively, with the weak and the weak* topology. Since $\mathcal{U}$ is reflexive here, its weak topology is also weak*, and thus the convergence in $\mathcal{Q}$ will be denoted simply by $\stackrel{*}{\rightharpoonup}$. We set $\mathsf{q}=(\mathsf{u},\mathsf{z})$, and we shall write, for instance, (t,q) in place of $(t,\mathsf{u},\mathsf{z})$. The *sets of stable states* $\mathcal{S}_\varepsilon(t)$ and $\mathcal{S}_{\varepsilon,\kappa}(t)$, for $t\in[0,T]$ and $\varepsilon\geq0$, are defined as

$$\mathcal{S}_\varepsilon(t) := \big\{\mathsf{q}{\in}\mathcal{Q};\ \forall\check{\mathsf{q}}{\in}\mathcal{Q}:\ \mathsf{E}_\varepsilon(t,\mathsf{q})<+\infty,\quad \mathsf{E}_\varepsilon(t,\mathsf{q})\ \leq\ \mathsf{E}_\varepsilon(t,\check{\mathsf{q}})+\mathsf{R}(\check{\mathsf{z}}{-}\mathsf{z})\big\}, \tag{5.1}$$

$$\mathcal{S}_{\varepsilon,\kappa}(t) := \big\{\mathsf{q}{\in}\mathcal{Q};\ \forall\check{\mathsf{q}}{\in}\mathcal{Q}:\ \mathsf{E}_{\varepsilon,\kappa}(t,\mathsf{q})<+\infty,\ \mathsf{E}_{\varepsilon,\kappa}(t,\mathsf{q})\leq\mathsf{E}_{\varepsilon,\kappa}(t,\check{\mathsf{q}})+\mathsf{R}(\check{\mathsf{z}}{-}\mathsf{z})\big\}. \tag{5.2}$$

We start by discussing the case in which the parameter $\kappa>0$ is chosen and fixed. Our results will be achieved by applying general abstract theorems proven in [12, 25, 30]. In what follows we write and check the assumptions needed to apply those theorems.

Let (ε_n) be a sequence such that $0<\varepsilon_n\to0$.

We shall say that a sequence $(t_n,\mathsf{q}_n)_{n\in\mathbb{N}}$ is a *stable sequence* with respect to $(\mathsf{E}_{\varepsilon_n})$ and $(\mathcal{S}_{\varepsilon_n})$ if

$$\mathsf{q}_n\in\mathcal{S}_{\varepsilon_n}(t_n)\quad\text{ and }\quad\sup_{n\in\mathbb{N}}\mathsf{E}_{\varepsilon_n}(t_n,\mathsf{q}_n)<+\infty. \tag{5.3}$$

Similarly, we say that $(t_n,\mathsf{q}_n)_{n\in\mathbb{N}}$ is a *stable sequence* with respect to $(\mathsf{E}_{\varepsilon_n,\kappa})$ and $(\mathcal{S}_{\varepsilon_n,\kappa})$ if

$$\mathsf{q}_n\in\mathcal{S}_{\varepsilon_n,\kappa}(t_n)\quad\text{ and }\quad\sup_{n\in\mathbb{N}}\mathsf{E}_{\varepsilon_n,\kappa}(t_n,\mathsf{q}_n)<+\infty. \tag{5.4}$$

Hence the notion of stable sequence depends on which sequence of functionals it is referred; in the sequel we omit to explicitly state this reference when it can be easily deduced from the context.

The dissipation distance $\mathcal{D}(\mathsf{z}_1,\mathsf{z}_2)=\mathsf{R}(\mathsf{z}_2-\mathsf{z}_1)$ satisfies the following properties, corresponding to [30, Formulas (2.2)–(2.4)].

Pseudo distance:

$$\mathcal{D}(\mathsf{z}_1,\mathsf{z}_1)=0 \;\text{ and }\; \mathcal{D}(\mathsf{z}_1,\mathsf{z}_3)\le\mathcal{D}(\mathsf{z}_1,\mathsf{z}_2)+\mathcal{D}(\mathsf{z}_2,\mathsf{z}_3) \;\text{ for any } \mathsf{z}_1,\mathsf{z}_2,\mathsf{z}_3\in\mathcal{Z}. \tag{5.5}$$

Lower semi-continuity of $\mathcal{D}$:

$$\mathcal{D}:\mathcal{Z}\times\mathcal{Z}\to[0,+\infty] \text{ is w}^*\text{-lower semi-continuous.} \tag{5.6}$$

Positivity of $\mathcal{D}$:

$$\begin{array}{l}\text{if a sequence } (\mathsf{z}_n) \text{ in } \mathcal{Z} \text{ and } \mathsf{z}\in\mathcal{Z} \text{ are such that}\\ \min\{\mathcal{D}(\mathsf{z}_n,\mathsf{z}),\mathcal{D}(\mathsf{z},\mathsf{z}_n)\}\to 0, \text{ then } \mathsf{z}_n\to\mathsf{z} \text{ weakly* in } \mathcal{Z}.\end{array} \tag{5.7}$$

From (5.6) it follows (2.5) of [30], that is

Lower Γ-limit for $\mathcal{D}$:

$$\begin{array}{l}\text{for any pair of stable sequences } (t_n,\mathsf{q}_n),\ (\check{t}_n,\check{\mathsf{q}}_n) \text{ such that}\\ (t_n,\mathsf{q}_n)\stackrel{*}{\rightharpoonup}(t,\mathsf{q}),\ (\check{t}_n,\check{\mathsf{q}}_n)\stackrel{*}{\rightharpoonup}(\check{t},\check{\mathsf{q}}) \text{ in } [0,T]\times\mathcal{Q}, \text{ we have}\\ \mathcal{D}(\mathsf{z},\check{\mathsf{z}})\le\liminf_{n\to+\infty}\mathcal{D}(\mathsf{z}_n,\check{\mathsf{z}}_n).\end{array} \tag{5.8}$$

This last property, involving stable sequences, depends, of course, on the sequence of problems under consideration.

For simplicity, from now on we continue by checking properties of the sequence $\mathsf{E}_{\varepsilon,\kappa}$ only, that is with the proof of the second "horizontal" convergence in (1.6); in fact, the proof of the first "horizontal" convergence in (1.6) can be done by following exactly the same steps and arguments.

From Korn's inequality, (2.3), (3.2) and (3.3), we find

$$\begin{aligned}\mathsf{E}_{\varepsilon,\kappa}(t,\mathsf{u},\mathsf{z})&\ge\frac{1}{2}\int_{\Omega_1\cup\Omega_2}\mathbb{C}e^{\varepsilon}(\mathsf{u}+\mathsf{w}(t)):e^{\varepsilon}(\mathsf{u}+\mathsf{w}(t))-2\mathsf{f}(t)\cdot(\mathsf{u}+\mathsf{w}(t))\,\mathrm{d}x\ge\\ &\ge c\|e(\mathsf{u}+\mathsf{w}(t))\|^2_{L^2}-\frac{1}{2\alpha}\|\mathsf{f}(t)\|^2_{L^2}-\frac{\alpha}{2}\|\mathsf{u}+\mathsf{w}(t)\|^2_{L^2}\ge K\|\mathsf{u}\|^2_{W^{1,2}}-C.\end{aligned} \tag{5.9}$$

This last inequality, together with a similar computation for $\mathsf{E}_{0,\kappa}$, shows that the set $\bigcup_{n\in\mathbb{N}}\{\mathsf{q}\in\mathcal{Q}:\ \mathsf{E}_{\varepsilon_n,\kappa}(t,\mathsf{q})\le E\}$ is weakly relatively compact. Moreover, for any $\varepsilon\ge0$ the functionals $\mathsf{E}_{\varepsilon,\kappa}(t,\cdot)$ are weakly lower semicontinuous in $\mathcal{Q}$ due to the convexity of the bulk part of the energy and Lemma 4.1. Thus the sublevels are also closed and the following property (corresponding to (2.6) of [30]) holds

Compactness of energy sublevels:

$$\begin{array}{l}\text{for all } t\in[0,T] \text{ and all } E\in\mathbb{R} \text{ we have}\\ (i)\ \{\mathsf{q}\in\mathcal{Q}:\ \mathsf{E}_{\varepsilon_n,\kappa}(t,\mathsf{q})\le E\} \text{ is compact for any } n\in\mathbb{N}\cup\{+\infty\};\\ (ii)\ \bigcup_{n\in\mathbb{N}\cup\{+\infty\}}\{\mathsf{q}\in\mathcal{Q}:\ \mathsf{E}_{\varepsilon_n,\kappa}(t,\mathsf{q})\le E\} \text{ is relatively compact.}\end{array} \tag{5.10}$$

Above and hereafter, to shorten the notation we shall set

$$\varepsilon_{+\infty}:=0.$$

Since f and w are continuously differentiable with respect to t (see Section 3), then $\mathsf{E}_{\varepsilon,\kappa}(\cdot,\mathsf{q})\in C^1([0,T])$ for all $\varepsilon\ge0$ and all $\mathsf{q}\in\mathcal{Q}$ for which $\mathsf{E}_{\varepsilon,\kappa}(\cdot,\mathsf{q})<+\infty$.

If $\mathsf{E}_{\varepsilon,\kappa}(s,\mathsf{u},\mathsf{z}) < +\infty$ for $s \in [0,T]$, then we have

$$\partial_t \mathsf{E}_{\varepsilon,\kappa}(t,\mathsf{u},\mathsf{z}) = \int_{\Omega_1\cup\Omega_2} \mathbb{C}e^\varepsilon(\dot{\mathsf{w}}(t)):e^\varepsilon(\mathsf{u}+\mathsf{w}(t)) - \dot{\mathsf{f}}(t)\cdot(\mathsf{u}+\mathsf{w}(t)) - \mathsf{f}(t)\cdot\dot{\mathsf{w}}(t)\,\mathrm{d}x \tag{5.11}$$

and, since $(e(\dot{\mathsf{w}}))_{i3} = 0$ (see (3.3)), by inequality (5.9) we have

$$\begin{aligned}|\partial_t \mathsf{E}_{\varepsilon,\kappa}(t,\mathsf{u},\mathsf{z})| &\le c\Big(\|e(\dot{\mathsf{w}})(t)\|^2_{L^2} + \|e^\varepsilon(\mathsf{u}+\mathsf{w}(t))\|^2_{L^2} + \|\dot{\mathsf{f}}(t)\|^2_{L^2} + \|\mathsf{u}\|^2_{L^2} + \|\mathsf{w}(t)\|^2_{L^2} + \|\mathsf{f}(t)\|^2_{L^2} + \|\dot{\mathsf{w}}(t)\|^2_{L^2}\Big)\\ &\le c\Big(\|e(\dot{\mathsf{w}})(t)\|^2_{L^2} + \|\dot{\mathsf{f}}(t)\|^2_{L^2} + \|\mathsf{w}(t)\|^2_{L^2} + \|\mathsf{f}(t)\|^2_{L^2} + \|\dot{\mathsf{w}}(t)\|^2_{L^2} + \mathsf{E}_{\varepsilon,\kappa}(t,\mathsf{u},\mathsf{z}) + C\Big),\end{aligned}$$

which, together with a similar computation for $\mathsf{E}_{0,\kappa}$, leads to (see (2.7) of [30])

Uniform control of the power $\partial_t\mathsf{E}_{\varepsilon,\kappa}$:

$$\begin{array}{l}\text{there exist } c_0^E \in \mathbb{R} \text{ and } c_1^E > 0 \text{ such that}\\ \text{for any } n \in \mathbb{N}\cup\{+\infty\},\ t\in[0,T] \text{ and } \mathsf{q}\in\mathcal{Q},\\ \text{if } \mathsf{E}_{\varepsilon_n,\kappa}(t,\mathsf{q}) < +\infty \text{ then } \mathsf{E}_{\varepsilon_n,\kappa}(\cdot,\mathsf{q}) \in C^1([0,T]) \text{ and}\\ |\partial_t \mathsf{E}_{\varepsilon_n,\kappa}(s,\mathsf{q})| \le c_1^E(c_0^E + \mathsf{E}_{\varepsilon_n,\kappa}(s,\mathsf{q})) \text{ for all } s\in[0,T].\end{array} \tag{5.12}$$

From the definition of $\mathsf{E}_{0,\kappa}$ (given in (4.5)) it follows that if $\mathsf{E}_{0,\kappa}(0,\mathsf{u},\mathsf{z})$ is finite then $\mathsf{u} \in W^{1,2}_{\mathrm{KL}}(\Omega_1\cup\Omega_2;\mathbb{R}^3)$ and thus $\mathsf{E}_{0,\kappa}(t,\mathsf{u},\mathsf{z})$ is finite for every $t\in[0,T]$. Since f and w are C^1 with respect to t, condition (2.8) of [30] is satisfied, namely

Uniform time-continuity of the power $\partial_t\mathsf{E}_{0,\kappa}$:

$$\begin{array}{l}\text{for every } \eta>0 \text{ and } E\in\mathbb{R} \text{ there exists } \delta>0 \text{ such that}\\ \mathsf{E}_{0,\kappa}(0,\mathsf{q}) \le E,\ |t_1-t_2|<\delta \Rightarrow |\partial_t\mathsf{E}_{0,\kappa}(t_1,\mathsf{q}) - \partial_t\mathsf{E}_{0,\kappa}(t_2,\mathsf{q})| < \eta.\end{array} \tag{5.13}$$

By Lemma 4.2 we get the following property (2.10) of [30]

Lower Γ-limit for $\mathsf{E}_{\varepsilon,\kappa}$:

$$\begin{array}{l}\text{for any sequence } 0<\varepsilon_n\to 0\\ \text{and any stable sequence } (t_n,\mathsf{q}_n) \text{ w.r. to } (\mathsf{E}_{\varepsilon_n,\kappa}) \text{ such that}\\ (t_n,\mathsf{q}_{\varepsilon_n}) \overset{*}{\rightharpoonup} (t,\mathsf{q}) \text{ in } [0,T]\times\mathcal{Q}, \text{ we have}\\ \mathsf{E}_{0,\kappa}(t,\mathsf{q}) \le \liminf_{n\to+\infty} \mathsf{E}_{\varepsilon_n,\kappa}(t_n,\mathsf{q}_{\varepsilon_n}).\end{array} \tag{5.14}$$

Remark 5.1 It is easy to check that the properties (5.10) – (5.14) hold also for the sequence E_ε with just natural and appropriate changes in the statements.

The following lemma essentially corresponds to property (2.9) of [30]; the only difference is that we establish the convergence in the open interval $(0,T)$ instead of its closure. This fact will not affect the arguments used in the sequel. The result was obtained in [12] for a single functional, while here we deal with a sequence of functionals.

Lemma 5.2 (Conditioned continuous convergence of the power.) *Let $0<\varepsilon_n\to 0$. Let $(t,\mathsf{q})\in(0,T)\times\mathcal{Q}$ and let (t_n,q_n) be a stable sequence with respect to $(\mathsf{E}_{\varepsilon_n,\kappa})$ and $(\mathcal{S}_{\varepsilon_n,\kappa})$. If $(t_n,\mathsf{q}_n) \overset{*}{\rightharpoonup} (t,\mathsf{q})$ in $[0,T]\times\mathcal{Q}$, then*

$$\partial_t\mathsf{E}_{\varepsilon_n,\kappa}(t_n,\mathsf{q}_n) \to \partial_t\mathsf{E}_{0,\kappa}(t,\mathsf{q}).$$

Proof: We combine the argument of the proof of [12, Proposition 3.3], where a single energy functional is considered, with the upper and lower bound Lemma 4.2 and Lemma 4.3.

Let $\mathsf{q}=(\mathsf{u},\mathsf{z})$. We start by showing that, for any $\kappa>0$,

$$\mathsf{E}_{\varepsilon_n,\kappa}(t_n,\mathsf{u}_n,\mathsf{z}_n) \to \mathsf{E}_{0,\kappa}(t,\mathsf{u},\mathsf{z}) \qquad \text{as} \quad n\to+\infty. \tag{5.15}$$

Indeed, from the stability of the sequence $(t_n, \mathsf{u}_n, \mathsf{z}_n)$ and by Lemma 4.3, there exist $\check{\mathsf{z}}_n \in L^\infty(\Gamma_{\mathrm{C}})$ with $0 \le \check{\mathsf{z}}_n \le 1$ a.e. on Γ_{C} and $\check{\mathsf{u}}_n \in W^{1,2}(\Omega_1 \cup \Omega_2; \mathbb{R}^3)$ such that $\check{\mathsf{u}}_n \rightharpoonup \mathsf{u}$ in $W^{1,2}(\omega_1 \cup \omega_2; \mathbb{R}^3)$, $\check{\mathsf{z}}_n \overset{*}{\rightharpoonup} \mathsf{z}$ in $L^\infty(\Gamma_{\mathrm{C}})$, such that

$$\limsup_{n\to+\infty} \mathsf{E}_{\varepsilon_n,\kappa}(t_n, \mathsf{u}_n, \mathsf{z}_n) \le \limsup_{n\to+\infty} \left[\mathsf{E}_{\varepsilon_n,\kappa}(t_n, \check{\mathsf{u}}_n, \check{\mathsf{z}}_n) + \mathsf{R}(\check{\mathsf{z}}_n - \mathsf{z}_n)\right] \le \mathsf{E}_{0,\kappa}(t, \mathsf{u}, \mathsf{z}).$$

Hence (5.15) follows from the inequality above and Lemma 4.2.

Let now $\delta > 0$ be such that $t \pm 2\delta \in [0, T]$. Then for any n large enough $t_n \in (t-\delta, t+\delta)$ and hence $t_n \pm \delta \in [0, T]$, and let $K_0 > 0$ be such that $\mathsf{E}_{\varepsilon_n,\kappa}(t_n, \mathsf{q}_n), \mathsf{E}_{0,\kappa}(t, \mathsf{q}) \le K_0$. By Korn's inequality we have

$$\begin{aligned}
&\Big|\frac{\mathsf{E}_{\varepsilon_n,\kappa}(t_n \pm \delta, \mathsf{u}_n, \mathsf{z}_n) - \mathsf{E}_{\varepsilon_n,\kappa}(t_n, \mathsf{u}_n, \mathsf{z}_n)}{\delta} \mp \partial_t \mathsf{E}_{\varepsilon_n,\kappa}(t_n, \mathsf{u}_n, \mathsf{z}_n)\Big| = \\
&\quad = \left|\partial_t \mathsf{E}_{\varepsilon_n,\kappa}(\bar{t}_n, \mathsf{u}_n, \mathsf{z}_n) - \partial_t \mathsf{E}_{\varepsilon_n,\kappa}(t_n, \mathsf{u}_n, \mathsf{z}_n)\right| \\
&\quad = \Big| \int_{\Omega_1 \cup \Omega_2} \mathbb{C} e^{\varepsilon_n}\big(\dot{\mathsf{w}}(\bar{t}_n)\big) : e^{\varepsilon_n}\big(\mathsf{w}(\bar{t}_n) - \mathsf{w}(t_n)\big) + \mathbb{C} e^{\varepsilon_n}\big(\dot{\mathsf{w}}(\bar{t}_n) - \dot{\mathsf{w}}(t_n)\big) : e^{\varepsilon_n}\big(\mathsf{u}_n + \mathsf{w}(t_n)\big) \\
&\qquad + \big(\dot{\mathsf{f}}(t_n) - \dot{\mathsf{f}}(\bar{t}_n)\big) \cdot \mathsf{u}_n + \dot{\mathsf{f}}(t_n) \cdot \mathsf{w}(t_n) - \dot{\mathsf{f}}(\bar{t}_n) \cdot \mathsf{w}(\bar{t}_n) + \mathsf{f}(\bar{t}_n) \cdot \dot{\mathsf{w}}(\bar{t}_n) - \mathsf{f}(t_n) \cdot \dot{\mathsf{w}}(t_n) \, \mathrm{d}x \Big| \\
&\quad \le C\Big(\|e(\dot{\mathsf{w}}(\bar{t}_n))\|_{L^2} \|e(\mathsf{w}(\bar{t}_n) - \mathsf{w}(t_n))\|_{L^2} + \|e(\dot{\mathsf{w}}(\bar{t}_n) - \dot{\mathsf{w}}(t_n))\|_{L^2} \|e^{\varepsilon_n}(\mathsf{u}_n + \mathsf{w}(t_n))\|_{L^2} \\
&\qquad + \|\dot{\mathsf{f}}(\bar{t}_n) - \dot{\mathsf{f}}(t_n)\|_{L^2} \|\mathsf{u}_n\|_{L^2} + \|\dot{\mathsf{f}}(t_n) \cdot \mathsf{w}(t_n) - \dot{\mathsf{f}}(\bar{t}_n) \cdot \mathsf{w}(\bar{t}_n)\|_{L^2} + \|\mathsf{f}(\bar{t}_n) \cdot \dot{\mathsf{w}}(\bar{t}_n) - \mathsf{f}(t_n) \cdot \dot{\mathsf{w}}(t_n)\|_{L^2} \Big) \\
&\quad \le C\Big(\|e(\dot{\mathsf{w}}(\bar{t}_n))\|_{L^2} \|e(\mathsf{w}(\bar{t}_n) - \mathsf{w}(t_n))\|_{L^2} + \|e(\dot{\mathsf{w}}(\bar{t}_n) - \dot{\mathsf{w}}(t_n))\|_{L^2} + \|\dot{\mathsf{f}}(\bar{t}_n) - \dot{\mathsf{f}}(t_n)\|_{L^2} \\
&\qquad + \|\dot{\mathsf{f}}(t_n) \cdot \mathsf{w}(t_n) - \dot{\mathsf{f}}(\bar{t}_n) \cdot \mathsf{w}(\bar{t}_n)\|_{L^2} + \|\mathsf{f}(\bar{t}_n) \cdot \dot{\mathsf{w}}(\bar{t}_n) - \mathsf{f}(t_n) \cdot \dot{\mathsf{w}}(t_n)\|_{L^2} \Big) K_0 =: \omega_{K_0}(\delta)
\end{aligned} \tag{5.16}$$

where $\bar{t}_n$ is between t_n and $t_n \pm \delta$, and where ω_{K_0} is independent of n because of the continuity at time t of w, $\dot{\mathsf{w}}$, f and $\dot{\mathsf{f}}$ stated in the assumptions at the beginning of Section 3.

By applying Lemma 4.2 and (5.15) we have

$$\liminf_{n\to+\infty} \frac{\mathsf{E}_{\varepsilon_n,\kappa}(t_n \pm \delta, \mathsf{u}_n, \mathsf{z}_n) - \mathsf{E}_{\varepsilon_n,\kappa}(t_n, \mathsf{u}_n, \mathsf{z}_n)}{\delta} \ge \frac{\mathsf{E}_{0,\kappa}(t \pm \delta, \mathsf{u}, \mathsf{z}) - \mathsf{E}_{0,\kappa}(t, \mathsf{u}, \mathsf{z})}{\delta}. \tag{5.17}$$

By using (5.16), then the inequality above and finally (5.13), we find

$$\begin{aligned}
\limsup_{n\to+\infty} \partial_t \mathsf{E}_{\varepsilon_n,\kappa}(t_n, \mathsf{u}_n, \mathsf{z}_n) &\le \limsup_{n\to+\infty} \frac{\mathsf{E}_{\varepsilon_n,\kappa}(t_n, \mathsf{u}_n, \mathsf{z}_n) - \mathsf{E}_{\varepsilon_n,\kappa}(t_n - \delta, \mathsf{u}_n, \mathsf{z}_n)}{\delta} + \omega_{K_0}(\delta) \\
&= \omega_{K_0}(\delta) - \liminf_{n\to+\infty} \frac{\mathsf{E}_{\varepsilon_n,\kappa}(t_n - \delta, \mathsf{u}_n, \mathsf{z}_n) - \mathsf{E}_{\varepsilon_n,\kappa}(t_n, \mathsf{u}_n, \mathsf{z}_n)}{\delta} \\
&\le \omega_{K_0}(\delta) - \frac{\mathsf{E}_{0,\kappa}(t - \delta, \mathsf{u}, \mathsf{z}) - \mathsf{E}_{0,\kappa}(t, \mathsf{u}, \mathsf{z})}{\delta} \le \partial_t \mathsf{E}_{0,\kappa}(t, \mathsf{u}, \mathsf{z}) + 2\omega_{K_0}(\delta).
\end{aligned}$$

In the same way, but appropriately choosing the signs in (5.16) and (5.17), we get

$$\liminf_{n\to+\infty} \partial_t \mathsf{E}_{\varepsilon_n,\kappa}(t_n, \mathsf{u}_n, \mathsf{z}_n) \ge \partial_t \mathsf{E}_{0,\kappa}(t, \mathsf{u}, \mathsf{z}) - 2\omega_{K_0}(\delta),$$

and the conclusion follows by letting δ go to zero.

With the same proof we can prove also the following analogous result concerning the sequence E_ε.

Lemma 5.3 (Conditioned continuous convergence of the power.) *Let $0 < \varepsilon_n \to 0$. Let $(t, \mathsf{q}) \in (0, T) \times \mathcal{Q}$ and let (t_n, q_n) be a stable sequence with respect to $(\mathsf{E}_{\varepsilon_n})$ and $(\mathcal{S}_{\varepsilon_n})$. If $(t_n, \mathsf{q}_n) \overset{*}{\rightharpoonup} (t, \mathsf{q})$ in $[0, T] \times \mathcal{Q}$, then*

$$\partial_t \mathsf{E}_{\varepsilon_n}(t_n, \mathsf{q}_n) \to \partial_t \mathsf{E}_0(t, \mathsf{q}).$$

By [30, Proposition 2.2] we get that Lemma 4.3 implies property (2.11) of [30], namely

Conditioned upper-semicontinuity of stable sets:

$$\begin{aligned}&\text{- for any stable sequence } (t_n, \mathsf{q}_n) \text{ w. r. to } (\mathsf{E}_{\varepsilon_n,\kappa}) \text{ and } (\mathcal{S}_{\varepsilon_n,\kappa}) \text{ such that}\\ &\quad (t_n, \mathsf{q}_n) \stackrel{*}{\rightharpoonup} (t, \mathsf{q}) \text{ in } [0,T]\times\mathcal{Q} \text{ we have that } \mathsf{q}\in\mathcal{S}_{0,\kappa}(t),\\ &\text{- for any stable sequence } (t_n, \mathsf{q}_n) \text{ w. r. to } (\mathsf{E}_{\varepsilon_n}) \text{ and } (\mathcal{S}_{\varepsilon_n}) \text{ such that}\\ &\quad (t_n, \mathsf{q}_n) \stackrel{*}{\rightharpoonup} (t, \mathsf{q}) \text{ in } [0,T]\times\mathcal{Q} \text{ we have that } \mathsf{q}\in\mathcal{S}_{0}(t).\end{aligned} \tag{5.18}$$

The existence of energetic solutions associated with the functionals E_ε, $\mathsf{E}_{\varepsilon,\kappa}$ and R, for fixed $\varepsilon > 0$, has been proven in [20]. In order to make the paper self-contained we give here a further proof based on a general existence theorem of Mielke [26], see also [12, 25, 30].

Theorem 5.4 (Existence of solution for ε fixed.)
(i) Let $\mathsf{q}^0_\kappa = (\mathsf{u}^0_\kappa, \mathsf{z}^0_\kappa) \in \mathcal{S}_{\varepsilon,\kappa}(0)$. *There exists an energetic solution* $(\mathsf{u}_\kappa, \mathsf{z}_\kappa):[0,T] \to \mathcal{Q}$ *to the problem* $(\mathcal{U}\times\mathcal{Z}, \mathsf{E}_{\varepsilon,\kappa}, \mathsf{R}, \mathsf{q}^0_\kappa)$.
(ii) Let $\mathsf{q}^0 = (\mathsf{u}^0, \mathsf{z}^0) \in \mathcal{S}_\varepsilon(0)$. *There exists an energetic solution* $(\mathsf{u}, \mathsf{z}) : [0,T] \to \mathcal{Q}$ *to the problem* $(\mathcal{U}\times\mathcal{Z}, \mathsf{E}_\varepsilon, \mathsf{R}, \mathsf{q}^0)$.

Proof: The proof follows by applying [26, Theorem 5.2]. Since we already know that the assumptions on the energy functional (5.10), (5.12), (5.13) with $\mathsf{E}_{0,\kappa}$ replaced by $\mathsf{E}_{\varepsilon,\kappa}$, and the conditions on the dissipation distance (5.5), (5.6), (5.7) hold true, to apply [26, Theorem 5.2], we only need to check the following compatibility conditions:

if (t_n, q_n) is a sequence such that $\mathsf{q}_n \in \mathcal{S}_\varepsilon(t_n)$ and $\sup_{n\in\mathbb{N}} \mathsf{E}_\varepsilon(t_n, \mathsf{q}_n) < +\infty$
(that is a so-called *stable sequence*)
and such that $(t_n, \mathsf{q}_n) \stackrel{*}{\rightharpoonup} (t, \mathsf{q})$ in $[0,T]\times\mathcal{Q}$, then

$$\partial_t \mathsf{E}_{\varepsilon,\kappa}(t_n, \mathsf{q}_n) \to \partial_t \mathsf{E}_{\varepsilon,\kappa}(t, \mathsf{q}) \quad \text{and} \quad \mathsf{q}\in\mathcal{S}_{\varepsilon,\kappa}(t).$$

The convergence of the powers follows from (5.11), using the time continuity of w, $\dot{\mathsf{w}}$, f and $\dot{\mathsf{f}}$. Passing to the stability condition $\mathsf{q} \in \mathcal{S}_{\varepsilon,\kappa}(t)$, setting $\mathsf{q} = (\mathsf{u}, \mathsf{z})$, we have to prove that

$$\mathsf{E}_{\varepsilon,\kappa}(t, \mathsf{u}, \mathsf{z}) \le \mathsf{E}_{\varepsilon,\kappa}(t, \check{\mathsf{u}}, \check{\mathsf{z}}) + \mathsf{R}(\check{\mathsf{z}}-\mathsf{z}) \quad \text{for every } (\check{\mathsf{u}}, \check{\mathsf{z}})\in\mathcal{Q}.$$

Let $(\check{\mathsf{u}}, \check{\mathsf{z}}) \in \mathcal{Q}$. Without loss of generality we may assume $(\check{\mathsf{u}}, \check{\mathsf{z}}) \in \mathcal{A}$ and $\check{\mathsf{z}} \le \mathsf{z}$ a.e. on $\Gamma_{\rm C}$. Let $\mathsf{q}_n = (\mathsf{u}_n, \mathsf{z}_n) \in \mathcal{S}_{\varepsilon,\kappa}(t_n)$ and choose the mutual recovery sequence $\check{\mathsf{q}}_n = (\check{\mathsf{u}}_n, \check{\mathsf{z}}_n)$ as $\check{\mathsf{u}}_n = \check{\mathsf{u}}$ and $\check{\mathsf{z}}_n$ as in (4.12). Then we have $\check{\mathsf{q}}_n \stackrel{*}{\rightharpoonup} \check{\mathsf{q}}$. Since, moreover, $\check{\mathsf{z}}_n - \mathsf{z}_n \le 0$ then

$$\mathsf{E}_{\varepsilon,\kappa}(t_n, \mathsf{u}_n, \mathsf{z}_n) \le \mathsf{E}_{\varepsilon,\kappa}(t_n, \check{\mathsf{u}}_n, \check{\mathsf{z}}_n) + \mathsf{R}(\check{\mathsf{z}}_n-\mathsf{z}_n),$$

and using the lower semicontinuity of $\mathsf{E}_{\varepsilon,\kappa}(t,\cdot)$ in $\mathcal{Q}$ together with the time regularity of w and f we deduce

$$\mathsf{E}_{\varepsilon,\kappa}(t, \mathsf{u}, \mathsf{z}) \le \liminf_{k\to+\infty} \mathsf{E}_{\varepsilon,\kappa}(t_n, \mathsf{u}_n, \mathsf{z}_n) \le \liminf_{k\to+\infty} \mathsf{E}_{\varepsilon,\kappa}(t_n, \check{\mathsf{u}}_n, \check{\mathsf{z}}_n) + \mathsf{R}(\check{\mathsf{z}}_n-\mathsf{z}_n) \le \mathsf{E}_{\varepsilon,\kappa}(t, \check{\mathsf{u}}, \check{\mathsf{z}}) + \mathsf{R}(\check{\mathsf{z}}-\mathsf{z}),$$

which concludes the proof of the first part of the statement, while the second can be proven simply substituting E_ε and $\mathcal{S}_\varepsilon$ in place of $\mathsf{E}_{\varepsilon,\kappa}$ and $\mathcal{S}_{\varepsilon,\kappa}$.

We are now in a position to state our main dimension reduction results. Indeed, the next theorems give a precise sense to the "horizontal" convergences in (1.6).

Theorem 5.5 (Convergence for $\varepsilon \to 0$, $\kappa < +\infty$ fixed.) *Let* $\mathsf{q}_0 = (\mathsf{u}_0, \mathsf{z}_0) \in \mathcal{Q}$ *and, for any* $\varepsilon > 0$, $\mathsf{q}_{0,\varepsilon} = (\mathsf{u}_{0,\varepsilon}, \mathsf{z}_{0,\varepsilon}) \in \mathcal{S}_{\varepsilon,\kappa}(0)$ *with* $\mathsf{q}_{0,\varepsilon} \stackrel{*}{\rightharpoonup} \mathsf{q}_0$ *and* $\mathsf{E}_{\varepsilon,\kappa}(0, \mathsf{q}_{0,\varepsilon}) \to \mathsf{E}_{0,\kappa}(0, \mathsf{q}_0)$ *as* $\varepsilon \to 0$, *let further* $\mathsf{q}_\varepsilon = (\mathsf{u}_\varepsilon, \mathsf{z}_\varepsilon) : [0,T] \to \mathcal{Q}$ *be* *an energetic solution to the problem* $(\mathcal{U}\times\mathcal{Z}, \mathsf{E}_{\varepsilon,\kappa}, \mathsf{R}, \mathsf{q}_{0,\varepsilon})$. *Then, there exist a sequence* (ε_n) *such that* $0 < \varepsilon_n \to 0$ *and* $\mathsf{q} = (\mathsf{u}, \mathsf{z}) : [0, T] \to \mathcal{Q}$ *such that*

$$\mathsf{E}_{\varepsilon_n,\kappa}(t, \mathsf{q}_{\varepsilon_n}(t)) \to \mathsf{E}_{0,\kappa}(t, \mathsf{q}(t)) \quad \text{for every } t \in [0, T], \tag{5.19a}$$
$$\mathsf{R}(\mathsf{z}_{\varepsilon_n}(t)-\mathsf{z}_{\varepsilon_n}(0)) \to \mathsf{R}(\mathsf{z}(t)-\mathsf{z}(0)) \quad \text{for every } t \in [0, T], \tag{5.19b}$$
$$\partial_t \mathsf{E}_{\varepsilon_n,\kappa}(\cdot, \mathsf{q}_{\varepsilon_n}(\cdot)) \to \partial_t \mathsf{E}_{0,\kappa}(\cdot, \mathsf{q}(\cdot)) \quad \text{in } L^1(0, T), \tag{5.19c}$$
$$\mathsf{z}_{\varepsilon_n}(t) \overset{*}{\rightharpoonup} \mathsf{z}(t) \quad \text{in } L^\infty(\Gamma_{\mathrm{C}}) \text{ for every } t \in [0, T], \tag{5.19d}$$
$$\mathsf{u}_{\varepsilon_n}(t) \rightharpoonup \mathsf{u}(t) \quad \text{in } W^{1,2}(\Omega_1\cup\Omega_2; \mathbb{R}^3) \text{ for every } t\in[0, T]. \tag{5.19e}$$

Moreover, any q *obtained by this way is an energetic solution to the problem* $(\mathcal{U}\times\mathcal{Z}, \mathsf{E}_{0,\kappa}, \mathsf{R}, \mathsf{q}_0)$.

Proof: By applying a Helly's type theorem (namely [30, Theorem A.1]) to the sequence (z_ε), we have that there exists a sequence (ε_n) and $\mathsf{z} \in L^\infty(\Gamma_{\mathrm{C}})$ such that

$$\mathsf{z}_{\varepsilon_n}(t) \overset{*}{\rightharpoonup} \mathsf{z}(t) \quad \text{in } L^\infty(\Gamma_{\mathrm{C}}) \text{ for every } t \in [0, T]. \tag{5.20}$$

We have thus proved (5.19d).

Since $t \mapsto (\mathsf{u}_{\varepsilon_n}(t), \mathsf{z}_{\varepsilon_n}(t))$ is an energetic solution and the power is controlled uniformly in ε_n, cf. (5.12), by a Gronwall-inequality argument, it can be shown (see [12, 25, 30]) that $\mathsf{E}_{\varepsilon_n,\kappa}(t, \mathsf{u}_{\varepsilon_n}(t), \mathsf{z}_{\varepsilon_n}(t))$ is bounded uniformly in n. Hence, from the uniform coercivity of $\mathsf{E}_{\varepsilon_n,\kappa}(t, \cdot, \cdot)$, by Korn's inequality (see (5.9)), $\mathsf{u}_{\varepsilon_n}(t)$ is uniformly bounded in $W^{1,2}(\Omega_1\cup\Omega_2; \mathbb{R}^3)$ for every $t \in [0, T]$. Thus, for every $t \in [0, T]$ there exists a subsequence (ε_{n^t}) such that

$$\mathsf{u}_{\varepsilon_{n^t}}(t) \rightharpoonup \mathsf{u}(t) \quad \text{in } W^{1,2}(\Omega_1\cup\Omega_2; \mathbb{R}^3), \tag{5.21}$$

and

$$\theta(t) := \limsup_{n\to+\infty} \partial_t \mathsf{E}_{\varepsilon_n,\kappa}(t, \mathsf{q}_{\varepsilon_n}(t)) = \lim_{n^t\to+\infty} \partial_t \mathsf{E}_{\varepsilon_{n^t},\kappa}(t, \mathsf{q}_{\varepsilon_{n^t}}(t)). \tag{5.22}$$

From (5.20), (5.21), (5.22) and Lemma 5.2 it follows that

$$\theta(t) = \partial_t \mathsf{E}_{0,\kappa}(t, \mathsf{u}(t), \mathsf{z}(t)), \tag{5.23}$$

from which (5.19c) follows. By [30, Theorem 3.1] and its proof we deduce that $\mathsf{q}(t) := (\mathsf{u}(t), \mathsf{z}(t))$ is an energetic solution to the problem $(\mathcal{U}\times\mathcal{Z}, \mathsf{E}_{0,\kappa}, \mathsf{R}, \mathsf{q}(0))$. By the stability inequality for the limit problem and the strict convexity of the map $\mathsf{u} \mapsto \mathsf{E}_{0,\kappa}(t, \mathsf{u}(t), \mathsf{z}(t))$, the function u is uniquely determined by z. Hence the convergence in (5.21) holds for the whole sequence ε_n, that is (5.19e).

Let us prove (5.19a). First of all we note that, by Lemma 4.2, we have

$$\liminf_{n\to+\infty} \mathsf{E}_{\varepsilon_n,\kappa}(t, \mathsf{u}_{\varepsilon_n}(t), \mathsf{z}_{\varepsilon_n}(t)) \geq \mathsf{E}_{0,\kappa}(t, \mathsf{u}(t), \mathsf{z}(t)). \tag{5.24}$$

By Lemma 4.3 there exist $\check{\mathsf{q}}_{\varepsilon_n} := (\check{\mathsf{u}}_{\varepsilon_n}, \check{\mathsf{z}}_{\varepsilon_n}) \in \mathcal{Q}$ such that

$$\check{\mathsf{q}}_{\varepsilon_n} \overset{*}{\rightharpoonup} \mathsf{q}(t) \quad \text{in } \mathcal{Q}$$

and

$$\limsup_{n\to+\infty} \left[\mathsf{E}_{\varepsilon_n,\kappa}(t, \check{\mathsf{u}}_{\varepsilon_n}, \check{\mathsf{z}}_{\varepsilon_n}) + \mathsf{R}(\check{\mathsf{z}}_{\varepsilon_n} - \mathsf{z}_{\varepsilon_n}(t))\right] \leq \mathsf{E}_{0,\kappa}(t, \mathsf{u}(t), \mathsf{z}(t)).$$

This inequality together with the stability condition imply

$$\limsup_{n\to+\infty} \mathsf{E}_{\varepsilon_n,\kappa}(t, \mathsf{u}_{\varepsilon_n}(t), \mathsf{z}_{\varepsilon_n}(t)) \leq \mathsf{E}_{0,\kappa}(t, \mathsf{u}(t), \mathsf{z}(t)), \tag{5.25}$$

which, together with (5.24), implies (5.19a).

Since $(\mathsf{u}_{\varepsilon_n}, \mathsf{z}_{\varepsilon_n})$ is an energetic solution, from the energy balance follows that $\mathrm{Diss}_{\mathsf{R}}(\mathsf{z}, [0,t]) < +\infty$. Hence, by definition, for any partition $\{t_j : j = 1, \ldots, N\}$ of $[0,t]$ we have

$$\sum_{j=1}^{N} \mathsf{R}(\mathsf{z}_{\varepsilon_n}(t_j) - \mathsf{z}_{\varepsilon_n}(t_{j-1})) < +\infty$$

which implies that the map $t \mapsto \mathsf{z}_{\varepsilon_n}(t)$ is non-increasing for the partial ordering "$\leq$ a.e.", hence $\mathsf{z}_{\varepsilon_n}(t) - \mathsf{z}_{\varepsilon_n}(0) \leq 0$ a.e. on Γ_{C}. Then, by (5.20) we have

$$\mathsf{R}(\mathsf{z}_{\varepsilon_n}(t) - \mathsf{z}_{\varepsilon_n}(0)) = -\int_{\Gamma_{\mathrm{C}}} \mathsf{a}(\mathsf{z}_{\varepsilon_n}(t) - \mathsf{z}_{\varepsilon_n}(0))\,\mathrm{d}\mathscr{H}^2 \to \mathsf{R}(\mathsf{z}(t) - \mathsf{z}(0))$$

that is (5.19b). ∎

Theorem 5.6 (Convergence for $\varepsilon \to 0$, $\kappa = +\infty$.) *Let $\mathsf{q}_0 = (\mathsf{u}_0, \mathsf{z}_0) \in \mathcal{Q}$ and, for any $\varepsilon > 0$, $\mathsf{q}_{0,\varepsilon} = (\mathsf{u}_{0,\varepsilon}, \mathsf{z}_{0,\varepsilon}) \in \mathcal{S}_\varepsilon(0)$ with $\mathsf{q}_{0,\varepsilon} \overset{*}{\rightharpoonup} \mathsf{q}_0$ and $\mathsf{E}_\varepsilon(0, \mathsf{q}_{0,\varepsilon}) \to \mathsf{E}_0(0, \mathsf{q}_0)$ as $\varepsilon \to 0$, let further $\mathsf{q}_\varepsilon = (\mathsf{u}_\varepsilon, \mathsf{z}_\varepsilon) : [0,T] \to \mathcal{Q}$ be an energetic solution to the problem $(\mathcal{U}\times\mathcal{Z}, \mathsf{E}_\varepsilon, \mathsf{R}, \mathsf{q}_{0,\varepsilon})$. Then, there exist a sequence (ε_n) such that $0 < \varepsilon_n \to 0$ and $\mathsf{q} = (\mathsf{u}, \mathsf{z}) : [0,T] \to \mathcal{Q}$ such that*

$$\mathsf{E}_{\varepsilon_n}(t, \mathsf{q}_{\varepsilon_n}(t)) \to \mathsf{E}_0(t, \mathsf{q}(t)) \qquad \textit{for every } t \in [0,T], \tag{5.26a}$$
$$\mathsf{R}(\mathsf{z}_{\varepsilon_n}(t) - \mathsf{z}_{\varepsilon_n}(0)) \to \mathsf{R}(\mathsf{z}(t) - \mathsf{z}(0)) \qquad \textit{for every } t \in [0,T], \tag{5.26b}$$
$$\partial_t \mathsf{E}_{\varepsilon_n}(\cdot, \mathsf{q}_{\varepsilon_n}(\cdot)) \to \partial_t \mathsf{E}_0(\cdot, \mathsf{q}(\cdot)) \qquad \textit{in } L^1(0,T), \tag{5.26c}$$
$$\mathsf{z}_{\varepsilon_n}(t) \overset{*}{\rightharpoonup} \mathsf{z}(t) \qquad \textit{in } L^\infty(\Gamma_{\mathrm{C}}) \textit{ for every } t \in [0,T], \tag{5.26d}$$
$$\mathsf{u}_{\varepsilon_n}(t) \rightharpoonup \mathsf{u}(t) \qquad \textit{in } W^{1,2}(\Omega_1 \cup \Omega_2; \mathbb{R}^3) \textit{ for every } t \in [0,T]. \tag{5.26e}$$

Moreover, any q obtained by this way is an energetic solution to $(\mathcal{U}\times\mathcal{Z}, \mathsf{E}_0, \mathsf{R}, \mathsf{q}_0)$.

Proof: The proof is a simple adaptation of the same arguments used in the proof of Theorem 5.5. ∎

The next proposition, which we state for completeness, justifies the vertical arrows in the diagram (1.6); this differs from Proposition 2.1 in the rescaling and in the statement for $\varepsilon = 0$.

Proposition 5.7 (Limit to brittle model, i.e. $\kappa \to +\infty$) *Let $\varepsilon \geq 0$ be fixed, and let $\mathsf{q}^0_\kappa = (\mathsf{u}^0_\kappa, \mathsf{z}^0_\kappa)$ be a sequence of initial conditions, i.e. $\mathsf{q}^0_\kappa \in \mathcal{S}_{\varepsilon,\kappa}(0)$ and, in particular, $0 \leq \mathsf{z}^0_\kappa \leq 1$. Assume moreover that, as κ goes to $+\infty$, $\mathsf{u}^0_\kappa \rightharpoonup \mathsf{u}^0$ in $W^{1,2}(\Omega_1 \cup \Omega_2; \mathbb{R}^3)$, $\mathsf{z}^0_\kappa \overset{*}{\rightharpoonup} \mathsf{z}^0$ in $L^\infty(\Gamma_{\mathrm{C}})$, and $\mathsf{E}_{\varepsilon,\kappa}(0, \mathsf{q}^0_\kappa) \to \mathsf{E}_\varepsilon(0, \mathsf{q}^0)$. Then*

$$(\mathcal{U}\times\mathcal{Z}, \mathsf{E}_{\varepsilon,\kappa}, \mathsf{R}, \mathsf{q}^0_\kappa) \xrightarrow{\kappa \to +\infty} (\mathcal{U}\times\mathcal{Z}, \mathsf{E}_\varepsilon, \mathsf{R}, \mathsf{q}^0)$$

in the following sense: having a sequence of energetic solutions $\{(\mathsf{u}_{\varepsilon,\kappa}, \mathsf{z}_{\varepsilon,\kappa})\}_{\kappa>0}$ to the problem $(\mathcal{U}\times\mathcal{Z}, \mathsf{E}_{\varepsilon,\kappa}, \mathsf{R}, \mathsf{q}^0_\kappa)$, there exists a subsequence and some $(\mathsf{u}_\varepsilon, \mathsf{z}_\varepsilon)$ such that

$$\mathsf{u}_{\varepsilon,\kappa}(t) \rightharpoonup \mathsf{u}_\varepsilon(t) \qquad \textit{in } W^{1,2}(\Omega_1 \cup \Omega_2; \mathbb{R}^3) \textit{ for all } t \in [0,T], \tag{5.27a}$$
$$\mathsf{z}_{\varepsilon,\kappa}(t) \overset{*}{\rightharpoonup} \mathsf{z}_\varepsilon(t) \qquad \textit{in } L^\infty(\Gamma_{\mathrm{C}}) \textit{ for all } t \in [0,T], \tag{5.27b}$$
$$\partial_t \mathsf{E}_{\varepsilon,\kappa}(\cdot, \mathsf{u}_{\varepsilon,\kappa}(\cdot), \mathsf{z}_{\varepsilon,\kappa}(\cdot)) \rightharpoonup \partial_t \mathsf{E}_\varepsilon(\cdot, \mathsf{u}_\varepsilon(\cdot), \mathsf{z}_\varepsilon(\cdot)) \qquad \textit{in } L^1(0,T). \tag{5.27c}$$

Moreover, each sequence $(\mathsf{u}_\varepsilon, \mathsf{z}_\varepsilon)$ obtained as a limit of such a selected subsequence is an energetic solution to the problem $(\mathcal{U}\times\mathcal{Z}, \mathsf{E}_\varepsilon, \mathsf{R}, \mathsf{q}^0)$.

For $\varepsilon > 0$ the proposition has been proven for the unscaled problem in [35]. The case $\varepsilon = 0$ can be treated similarly.

6 Cracking Kirchhoff-Love plate reformulated

The aim of this section is to provide a two-dimensional formulation of the limit problems.

Let us recall that $\mathsf{u} \in W^{1,2}_{\rm KL}(\Omega_1{\cup}\Omega_2;\mathbb{R}^3)$ if and only if there exist $\rho = (\rho_1,\rho_2) \in W^{1,2}(\omega_1{\cup}\omega_2;\mathbb{R}^2)$ and $\xi \in W^{2,2}(\omega_1{\cup}\omega_2)$ such that

$$\begin{aligned}&\mathsf{u}_\alpha(x_1,x_2,x_3) = \rho_\alpha(x_1,x_2) - x_3\tfrac{\partial\xi}{\partial x_\alpha}(x_1,x_2) \qquad \text{for } \alpha=1,2, \text{ and}\\ &\mathsf{u}_3(x_1,x_2) = \xi(x_1,x_2)\end{aligned}$$

for $(x_1,x_2) \in \omega_1\cup\omega_2$ and $x_3 \in (-\frac{h}{2},\frac{h}{2})$; see Le Dret [23, Lemma 4.2]. Since the effective domains of the limit energies E_0 and $\mathsf{E}_{0,\kappa}$ are contained in the set of Kirchhoff-Love displacements $W^{1,2}_{\rm KL}(\Omega_1{\cup}\Omega_2;\mathbb{R}^3)$ from (4.3), it is thus possible to rewrite the limit energy functionals in terms of the Kirchhoff-Love generalized displacements ρ and ξ.

Recalling the expression (3.2) of w, we first observe that

$$\tilde e(\mathsf{u}) = \tilde e(\rho) - x_3\nabla^2\xi, \qquad \tilde e(\mathsf{w}) = \tilde e(\eta) - x_3\nabla^2\zeta$$

for some η and ζ given. In terms of these new variables, we have

$$\frac12\int_{\Omega_1\cup\Omega_2}\mathbb{C}^0\tilde e(\mathsf{u}{+}\mathsf{w}){:}\tilde e(\mathsf{u}{+}\mathsf{w})\,\mathrm{d}x = \frac12\int_{\omega_1\cup\omega_2} h\mathbb{C}^0\tilde e(\rho{+}\eta){:}\tilde e(\rho{+}\eta) + \frac{h^3}{12}\mathbb{C}^0\nabla^2(\xi{+}\zeta){:}\nabla^2(\xi{+}\zeta)\,\mathrm{d}\mathscr{H}^2$$

while

$$\begin{aligned}&\int_{\Omega_1\cup\Omega_2}\mathsf{f}{\cdot}(\mathsf{u}{+}\mathsf{w})\,\mathrm{d}x = \int_{\omega_1\cup\omega_2}\varphi^0_3(\xi{+}\zeta) + \sum_{\alpha=1,2}\varphi^0_\alpha(\rho_\alpha{+}\eta_\alpha) - \varphi^1_\alpha\tfrac{\partial}{\partial x_\alpha}(\xi{+}\zeta)\,\mathrm{d}\mathscr{H}^2\\ &\qquad\text{with}\quad \varphi^i_\alpha(x_1,x_2) := \int_{-h/2}^{h/2} x_3^i\mathsf{f}_\alpha(x_1,x_2,x_3)\,\mathrm{d}x_3, \qquad (x_1,x_2)\in\omega_1\cup\omega_2,\end{aligned} \tag{6.1}$$

where, for simplicity, we have not stressed the dependence on t of w, f, η, ζ, and φ^i_α. Since

$$\begin{aligned}\big|[\![\mathsf{u}]\!]_{\Gamma_{\rm C}}\big|^2 &= \big|[\![(\rho_1{-}x_3\tfrac{\partial\xi}{\partial x_1},\rho_2{-}x_3\tfrac{\partial\xi}{\partial x_2},\xi)]\!]_{\Gamma_{\rm C}}\big|^2\\ &= \big|[\![\rho]\!]_{\gamma_{\rm C}}\big|^2 + x_3^2\big|[\![\nabla\xi]\!]_{\gamma_{\rm C}}\big|^2 - 2x_3[\![\rho]\!]_{\gamma_{\rm C}}{\cdot}[\![\nabla\xi]\!]_{\gamma_{\rm C}} + [\![\xi]\!]^2_{\gamma_{\rm C}}\end{aligned} \tag{6.2}$$

we have that

$$\begin{aligned}\int_{\Gamma_{\rm C}}\mathsf{z}\kappa\big|[\![\mathsf{u}]\!]_{\Gamma_{\rm C}}\big|^2\,\mathrm{d}\mathscr{H}^2 &= \kappa\int_{\gamma_{\rm C}}\int_{-h/2}^{h/2}\mathsf{z}\big|[\![\mathsf{u}]\!]_{\Gamma_{\rm C}}\big|^2\,\mathrm{d}x_3\,\mathrm{d}\mathscr{H}^1\\ &= \kappa\int_{\gamma_{\rm C}} m_0(\mathsf{z})\big(\big|[\![\rho]\!]_{\gamma_{\rm C}}\big|^2 + [\![\xi]\!]^2_{\gamma_{\rm C}}\big) - 2m_1(\mathsf{z})[\![\rho]\!]_{\gamma_{\rm C}}{\cdot}[\![\nabla\xi]\!]_{\gamma_{\rm C}} + m_2(\mathsf{z})\big|[\![\nabla\xi]\!]_{\gamma_{\rm C}}\big|^2\,\mathrm{d}\mathscr{H}^1\end{aligned} \tag{6.3}$$

where we have introduced the algebraic momenta $m_i(\mathsf{z})$ defined by

$$m_i(\mathsf{z})(x_1,x_2) := \int_{-h/2}^{h/2} x_3^i\mathsf{z}(x_1,x_2,x_3)\,\mathrm{d}x_3, \qquad (x_1,x_2)\in\gamma_{\rm C}, \quad i\in\mathbb{N}. \tag{6.4}$$

Assuming from now on that a is independent of x_3, also the dissipation $\mathsf{R}(\tilde{\mathsf{z}}-\mathsf{z})$ can be expressed in terms of these momenta, namely

$$\mathsf{R}(\mathsf{z}-\tilde{\mathsf{z}}) = \int_{\gamma_{\rm C}}\int_{-h/2}^{h/2}\mathsf{a}(\tilde{\mathsf{z}}{-}\mathsf{z})\,\mathrm{d}x_3\mathrm{d}\mathscr{H}^1 = \int_{\gamma_{\rm C}}\mathsf{a}\big(m_0(\tilde{\mathsf{z}}){-}m_0(\mathsf{z})\big)\,\mathrm{d}\mathscr{H}^1 \tag{6.5}$$

provided $\mathsf{z}-\tilde{\mathsf{z}}\leq 0$ a.e. on $\Gamma_{\rm C}$, otherwise $\mathsf{R}(\mathsf{z}-\tilde{\mathsf{z}})=+\infty$. Concerning the adhesive model we have

$$\begin{aligned}\widehat{\mathsf{A}}_{\rm KL}^{\rm ad} := \Big\{&(\rho,\xi,\mathsf{z})\in W^{1,2}(\omega_1\cup\omega_2;\mathbb{R}^2)\times W^{2,2}(\omega_1\cup\omega_2)\times L^\infty(\Gamma_{\rm C}) \ : \\ &0\leq \mathsf{z}\leq 1 \text{ a.e. on } \Gamma_{\rm C},\ \big[\!\big[(\rho-x_3\nabla\xi)\cdot\nu\big]\!\big]_{\gamma_{\rm C}}\geq 0 \text{ for } x_3=\pm h/2 \text{ a.e. on } \gamma_{\rm C}, \\ &\rho|_{\gamma_{\rm Dir}}=\xi|_{\gamma_{\rm Dir}}=\nabla\xi\cdot\nu|_{\gamma_{\rm Dir}}=0 \text{ a.e. on } \gamma_{\rm Dir}\Big\},\end{aligned}\tag{6.6}$$

and the stored energy is

$$\widehat{\mathsf{E}}_{0,\kappa}(t,\rho,\xi,\mathsf{z}) := \begin{cases}\dfrac{1}{2}\displaystyle\int_{\omega_1\cup\omega_2} h\mathbb{C}^0\tilde{e}(\rho{+}\eta){:}\tilde{e}(\rho{+}\eta)+\frac{h^3}{12}\mathbb{C}^0\nabla^2(\xi{+}\zeta){:}\nabla^2(\xi{+}\zeta) \\ \qquad\qquad -\varphi_3^0(\xi{+}\zeta)-\sum_{\alpha=1,2}\varphi_\alpha^0(\rho_\alpha{+}\eta_\alpha)+\varphi_\alpha^1\frac{\partial}{\partial x_\alpha}(\xi{+}\zeta)\,\mathrm{d}\mathscr{H}^2 \\ +\kappa\displaystyle\int_{\gamma_{\rm C}} m_0\big(\big|[\![\rho]\!]_{\gamma_{\rm C}}\big|^2+[\![\xi]\!]_{\gamma_{\rm C}}^2\big)-2m_1[\![\rho]\!]_{\gamma_{\rm C}}\cdot[\![\nabla\xi]\!]_{\gamma_{\rm C}}+m_2\big|[\![\nabla\xi]\!]_{\gamma_{\rm C}}\big|^2\,\mathrm{d}\mathscr{H}^1 & \text{if } (\rho,\xi,\mathsf{z})\in\widehat{\mathsf{A}}_{\rm KL}^{\rm ad} \\ +\infty & \text{else.}\end{cases}$$

As to the brittle model we observe that $\mathsf{z}[\![\mathsf{u}]\!]_{\Gamma_{\rm C}}=0$ a.e. on $\Gamma_{\rm C}$ if and only if $\mathsf{z}\big|[\![\mathsf{u}]\!]_{\Gamma_{\rm C}}\big|^2=0$, and recalling the expression (6.2) we have

$$\mathsf{z}\big([\![\xi]\!]_{\gamma_{\rm C}}^2+\big|[\![\rho]\!]_{\gamma_{\rm C}}-x_3[\![\nabla\xi]\!]_{\gamma_{\rm C}}\big|^2\big)=0 \text{ a.e. on } \Gamma_{\rm C}.$$

Therefore, since the polynomial $x_3\mapsto\big|[\![\rho]\!]_{\gamma_{\rm C}}-x_3[\![\nabla\xi]\!]_{\gamma_{\rm C}}\big|^2$ identically vanishes if and only if its coefficients vanish a.e. on $\Gamma_{\rm C}$, we have

$$\widehat{\mathsf{A}}_{\rm KL} := \Big\{(\rho,\xi,\mathsf{z})\in\widehat{\mathsf{A}}_{\rm KL}^{\rm ad}:\ \ \mathsf{z}\equiv 0 \ \text{ or } \ [\![\rho]\!]_{\gamma_{\rm C}}=[\![\nabla\xi]\!]_{\gamma_{\rm C}}=[\![\xi]\!]_{\gamma_{\rm C}}=0 \text{ a.e. on } \Gamma_{\rm C}\Big\}$$

where $\widehat{\mathsf{A}}_{\rm KL}^{\rm ad}$ has been defined in (6.6). This clearly shows that in the regions where $\mathsf{z}>0$ the displacements have no jumps at all. Setting

$$\begin{aligned}\mathscr{E}_0(t,\rho,\xi) := \frac{1}{2}\int_{\omega_1\cup\omega_2} &h\mathbb{C}^0\tilde{e}(\rho{+}\eta){:}\tilde{e}(\rho{+}\eta)+\frac{h^3}{12}\mathbb{C}^0\nabla^2(\xi{+}\zeta){:}\nabla^2(\xi{+}\zeta) \\ &-\varphi_3^0(\xi{+}\zeta)-\sum_{\alpha=1,2}\varphi_\alpha^0(\rho_\alpha{+}\eta_\alpha)+\varphi_\alpha^1\frac{\partial}{\partial x_\alpha}(\xi{+}\zeta)\,\mathrm{d}\mathscr{H}^2,\end{aligned}\tag{6.7}$$

the stored energy writes as

$$\widehat{\mathsf{E}}_0(t,\rho,\xi,\mathsf{z}) := \begin{cases}\mathscr{E}_0(t,\rho,\xi) & \text{if } (\rho,\xi,\mathsf{z})\in\widehat{\mathsf{A}}_{\rm KL} \\ +\infty & \text{else.}\end{cases}$$

The evolution problem associated with $\widehat{\mathsf{E}}_0$ and R admits an alternative formulation governed by the following stored and dissipation energy functionals

$$\widehat{\mathsf{E}}_0^{\rm 2D}(t,\rho,\xi,\vartheta) := \begin{cases}\mathscr{E}_0(t,\rho,\xi) & \text{if } (\rho,\xi,\vartheta)\in\widehat{\mathsf{A}}_{\rm KL}^{\rm 2D}, \\ +\infty & \text{else,}\end{cases}$$

$$\mathsf{R}^{\rm 2D}(\dot\vartheta) := \begin{cases}h\displaystyle\int_{\gamma_{\rm C}}\mathsf{a}|\dot\vartheta|\,\mathrm{d}\mathscr{H}^2 & \text{if } \dot\vartheta\leq 0 \text{ on } \gamma_{\rm C}, \\ +\infty & \text{else,}\end{cases}$$

where

$$\begin{aligned}\widehat{\mathsf{A}}_{\rm KL}^{\rm 2D} := \Big\{&(\rho,\xi,\vartheta)\in W^{1,2}(\omega_1\cup\omega_2;\mathbb{R}^2)\times W^{2,2}(\omega_1\cup\omega_2)\times L^\infty(\gamma_{\rm C}):\ \ 0\leq\vartheta\leq 1 \ \text{ a.e. on } \gamma_{\rm C}, \\ &[\![(\rho-x_3\nabla\xi)\cdot\nu]\!]_{\gamma_{\rm C}}\geq 0 \ \text{ for } x_3=\pm h/2 \ \text{ a.e. on } \gamma_{\rm C}, \\ &\rho|_{\gamma_{\rm Dir}}=\xi|_{\gamma_{\rm Dir}}=\nabla\cdot\nu|_{\gamma_{\rm Dir}}=0 \ \text{ a.e. on } \gamma_{\rm Dir}, \\ &\vartheta=0 \ \text{ or } \ [\![\rho]\!]_{\gamma_{\rm C}}=[\![\nabla\xi]\!]_{\gamma_{\rm C}}=[\![\xi]\!]_{\gamma_{\rm C}}=0 \ \text{ a.e. on } \gamma_{\rm C}\Big\}.\end{aligned}$$

Indeed, it is easy to see that, if $t \mapsto \big(\rho(t), \xi(t), \mathsf{z}(t)\big)$ is an energetic solution associated with $\widehat{\mathsf{E}}_0$ and R then, letting

$$\vartheta(t, x_1, x_2) := \frac{m_0\big(\mathsf{z}(t, x_1, x_2, \cdot)\big)}{h}, \tag{6.8}$$

the map $t \mapsto \big(\rho(t), \xi(t), \vartheta(t)\big)$ is an energetic solution associated with $\widehat{\mathsf{E}}_0^{\rm 2D}$ and $\mathsf{R}^{\rm 2D}$. We observe that, in the case of brittle delamination, the limit problem thus admits a purely 2D formulation based on $\widehat{\mathsf{E}}_0^{\rm 2D}$ and $\mathsf{R}^{\rm 2D}$. Moreover, having an energetic solution $t \mapsto \big(\rho(t), \xi(t), \vartheta(t)\big)$, we can reconstruct an energetic solution (ρ, ξ, z) associated with $\widehat{\mathsf{E}}_0$ and R. The simplest way is by choosing $\mathsf{z}(t, x_1, x_2, \cdot)$ constant, i.e.

$$\mathsf{z}(t, x_1, x_2, x_3) := \vartheta(t, x_1, x_2). \tag{6.9}$$

Remark 6.1 (Conceptual numerical strategy.) After a time discretisation one obtains an incremental problem which might be used, on one hand, as a theoretical tool to prove existence and, on the other hand, to find a numerical solution (after further spatial discretisation and implementing a suitable global minimization strategy), see e.g. [28, 34]. Namely, considering a time step $\tau > 0$, the approximate solution obtained by the implicit time discretisation gives rise to the incremental minimisation problem for $\mathsf{E}_{0,\kappa}(k\tau, \mathsf{u}, \mathsf{z}) + \mathsf{R}(\mathsf{z}{-}\mathsf{z}_\tau^{k-1})$ subject to $(\mathsf{u}, \mathsf{z}) \in \mathcal{U}{\times}\mathcal{Z}$ whose solution is denoted by $(\mathsf{u}_\tau^k, \mathsf{z}_\tau^k)$ and used in the recursive scheme with $k = 1, ..., T/\tau$. Setting, for any $m = (m_0, m_1, m_2) \in L^\infty(\gamma_{\rm C}; \mathbb{R}^3)$,

$$\begin{aligned}\mathscr{E}_{0,\kappa}(t, \rho, \xi, m) :=& \frac{1}{2}\int_{\omega_1\cup\omega_2} h\mathbb{C}^0\tilde{e}(\rho{+}\eta){:}\tilde{e}(\rho{+}\eta) + \frac{h^3}{12}\mathbb{C}^0\nabla^2(\xi{+}\zeta){:}\nabla^2(\xi{+}\zeta)\\ &\qquad - \varphi_3^0(\xi{+}\zeta) - \sum_{\alpha=1,2}\varphi_\alpha^0(\rho_\alpha{+}\eta_\alpha) + \varphi_\alpha^1\frac{\partial}{\partial x_\alpha}(\xi{+}\zeta)\,\mathrm{d}\mathscr{H}^2\\ &+ \kappa\int_{\gamma_{\rm C}} m_0\big(\big|[\![\rho]\!]_{\gamma_{\rm C}}\big|^2 + [\![\xi]\!]_{\gamma_{\rm C}}^2\big) - 2m_1[\![\rho]\!]_{\gamma_{\rm C}}{\cdot}[\![\nabla\xi]\!]_{\gamma_{\rm C}} + m_2\big|[\![\nabla\xi]\!]_{\gamma_{\rm C}}\big|^2\,\mathrm{d}\mathscr{H}^1,\end{aligned}$$

the mentioned incremental problem reads as

$$\left.\begin{array}{ll}\text{Minimise} & \mathscr{E}_{0,\kappa}(k\tau, \rho, \xi, m(\mathsf{z})) + \displaystyle\int_{\gamma_{\rm C}} \mathsf{a}\big(m_0(\mathsf{z}){-}m_0(\mathsf{z}_\tau^{k-1})\big)\,\mathrm{d}\mathscr{H}^1,\\ \text{subject to} & \mathsf{z} \le \mathsf{z}_\tau^{k-1},\ \ (\rho, \xi, \mathsf{z}) \in \widehat{\mathsf{A}}_{\rm KL}^{\rm ad},\end{array}\right\} \tag{6.10}$$

for a given $0 \le \mathsf{z}_\tau^0 \le 1$ and for $m(\mathsf{z})$ from (6.4). In the adhesive case, it does not seem straightforward to translate the constraint $\mathsf{z} \le \tilde{\mathsf{z}}$ on $\Gamma_{\rm C}$ in terms of the momenta defined on $\gamma_{\rm C}$ and therefore the incremental problem (and the corresponding numerical strategy) should rather deal with the 2D generalised displacements and the original 2D profile z rather than its 1D momenta.

Remark 6.2 (The brittle case.) In the brittle delamination case, the situation is better and we can work in terms of the 1D profile ϑ rather than z. Using the functionals $\widehat{\mathsf{E}}_0^{\rm 2D}$ and $\mathsf{R}^{\rm 2D}$, the incremental brittle delamination limit problem reads as

$$\left.\begin{array}{ll}\text{Minimise} & \mathscr{E}_0(k\tau, \rho, \xi) + h\displaystyle\int_{\gamma_{\rm C}} \mathsf{a}(\vartheta{-}\vartheta_\tau^{k-1})\,\mathrm{d}\mathscr{H}^1,\\ \text{subject to} & 0 \le \vartheta \le \vartheta_\tau^{k-1}\ \text{ and }\ [\![(\rho{-}x_3\nabla\xi){\cdot}\nu]\!]_{\gamma_{\rm C}} \ge 0\ \text{ for }\ x_3 = \pm h/2\ \text{ a.e. on } \gamma_{\rm C},\\ & \rho|_{\gamma_{\rm Dir}} = \xi|_{\gamma_{\rm Dir}} = \nabla\xi\cdot\nu|_{\gamma_{\rm Dir}} = 0\ \text{ a.e. on } \gamma_{\rm Dir},\\ & \vartheta = 0\ \text{ or }\ [\![\rho]\!]_{\gamma_{\rm C}} = [\![\nabla\xi]\!]_{\gamma_{\rm C}} = [\![\xi]\!]_{\gamma_{\rm C}} = 0 \text{ a.e. on } \gamma_{\rm C},\\ & (\rho, \xi, \vartheta) \in W^{1,2}(\omega_1{\cup}\omega_2; \mathbb{R}^2) \times W^{2,2}(\omega_1{\cup}\omega_2) \times L^\infty(\gamma_{\rm C}),\end{array}\right\} \tag{6.11}$$

for a given $0 \le \vartheta_\tau^0 \le 1$. Although the dimensionality of (6.11) is lower than (6.10), the direct numerical implementation of the "or" structure appearing in the constraint in (6.11) is expected to be difficult. To circumvent this difficulty, one may implement a sequence of problems (6.10) with κ gradually increasing and rely on the analysis presented above, namely Proposition 5.7 for $\varepsilon = 0$. An interesting question is whether one can augment (6.10) with the additional constraint (6.9) without destroying the limit for $\kappa \to \infty$ and then, in terms of ϑ, implement efficiently such a lower-dimensional approximate variant of (6.10).

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