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Saint-Venant's theory for beams with multi-connected cross-section: justification and error estimate

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Abstract

The aim of the paper is twofold. First, starting from the three-dimensional theory of linear elasticity, we give a simple justification of Saint-Venant theory for beams with multi-connected cross-section by means of Γ-convergence. Second, we estimate the error between the three-dimensional problem and the limit problem.

1 Introduction

In the last years there has been a growing interest in the justification of the classical theories of thin-elastic bodies. Several methods have been used, among which the most popular are: the asymptotic expansion method, the Γ-convergence, and the use of functional analysis techniques similar in spirit to those applied in the mathematical theory of homogenization. Among these, the first provides only a formal justification since, as a starting point,

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it assumes the existence of a power series expansion of the unknowns in terms of the parameters which identify the small scales.

The justification by means of Γ-convergence of St. Venant beam's theory has been given for the first time by Anzellotti, Baldo and Percivale [1]. These authors limited themselves to consider circular cross sections, while the case of a general simply-connected cross section has been considered by Percivale [16]. Since then, several extensions have been given. In particular we mention the works by Mora and Muller [12], [13] where the starting point is non-linear elasticity instead of linear elasticity, and the studies of Freddi, Morassi and Paroni [4], [5] on thin-walled beams.

The method based on functional analysis techniques has been used by Le Dret [10] to justify the theory of rods. In this paper the author does not only study the convergence of the displacements but also of the stresses. The cross section is again assumed simply-connected and the material is homogeneous and isotropic. The extension to fully anisotropic inhomogeneous materials has been given by Murat and Sili [14], [15]. The method has been used also by the authors [6] to study anisotropic, inhomogeneous, thin-walled beams with rectangular cross-section.

In this paper we use the theory of Γ-convergence to give a rigorous justification of the linear theory of elastic beams with multi-connected cross-section which, to the best of our knowledge, has been done only through the formal asymptotic method (see for instance the paper by Trabucho and Viano [17]).

The key result which allows us to handle multi-connected cross-sections is contained in Theorem 3.3. The corresponding step in the simply-connected case is easily achieved thanks to an application of Poincaré's lemma, see [3]. Successively, we give an estimate of the error between the solutions of the three-dimensional problem and of the limit problem. This, to our knowledge, is the first rigorous error estimate of a dimension reduction problem given within the framework of Γ-convergence. Our estimate is in agreement with those obtained, in different frameworks, by Irago and Viaño, [8], and by Monneau, Murat, and Sili, [11].

We limit ourselves to the case of homogeneous and isotropic bodies to keep notation and proofs as simple as possible, even if the same proof holds for more general symmetries of the body. Moreover, since the problem is linear it could be studied by using the weak form of the problem following mainly the same ideas used in our Γ-convergence proof.

Unless otherwise stated, we use the Einstein summation convention and we index vector and tensor components as follows: Greek indices α, β take values in the set $\{1,2\}$ and Latin indices i, j in the set $\{1,2,3\}$. Convergence in the norm will be denoted by $\rightarrow$ while weak convergence will be denoted by $\rightharpoonup$. With a little, but harmless, abuse of notation, we shall call "sequences" families with index a continuous parameter ε which, throughout the paper, will be assumed to belong to the interval $(0,1]$.

2 The 3-dimensional problem

Let $\omega = \omega_0 \setminus \cup_{i=1}^{I} \overline{\omega_i} \subset \mathbb{R}^2$, where ω_j, for $j = 0, 1, \ldots, I$, are open, bounded, simply-connected sets with Lipschitz boundaries $\gamma_j := \partial\omega_j$. Moreover, we assume that $\overline{\omega_i} \subset \omega_0$, and $\overline{\omega_i} \cap \overline{\omega_k} = \varnothing$ for $i, k = 1, \ldots, I$, with $i \neq k$. Thus ω is a bounded, open, connected, possibly multi-connected set with a Lipschitz boundary, see Fig. 1. We assume that the coordinate axes are such that

$$\int_\omega x_1 \, dx_1 dx_2 = \int_\omega x_2 \, dx_1 dx_2 = \int_\omega x_1 x_2 \, dx_1 dx_2 = 0. \tag{1}$$

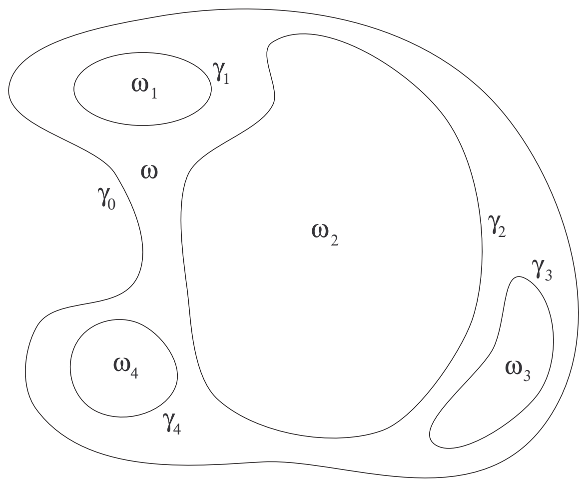

Fig. 1. The cross-section of the beam with $I = 4$.

For $\varepsilon \in (0,1]$, let $\omega_\varepsilon := \varepsilon\omega$, and for $\ell > 0$, let $\Omega_\varepsilon := \omega_\varepsilon \times (0,\ell) \subset \mathbb{R}^3$.

Henceforth, we shall refer to Ω_ε as the reference configuration occupied by a homogeneous, isotropic, linearly elastic body. We assume the body to be subject to body forces $b^\varepsilon \in L^2(\Omega_\varepsilon;\mathbb{R}^3)$ and to have null displacement on $\omega_\varepsilon \times \{x_3 = 0\}$. Then, the displacement may be computed by minimizing the energy functional

$$\mathcal{F}_\varepsilon(\tilde{u}) := \frac{1}{2}\int_{\Omega_\varepsilon} \mathbb{C}E\tilde{u}\cdot E\tilde{u}\,dx - \int_{\Omega_\varepsilon} b^\varepsilon\cdot\tilde{u}\,dx$$

among all displacements $\tilde{u} \in H^1_{dn}(\Omega_\varepsilon;\mathbb{R}^3)$, where

$$H^1_{dn}(\Omega_\varepsilon;\mathbb{R}^n) := \{\tilde{v} \in H^1(\Omega_\varepsilon;\mathbb{R}^n) : \tilde{v} = 0 \text{ on } \omega_\varepsilon \times \{x_3 = 0\}\}.$$

Above, $E\tilde{u}$ denotes the strain of the displacement $\tilde{u}$, i.e.,

$$E\tilde{u}(x) := \frac{D\tilde{u}(x) + D\tilde{u}^\mathsf{T}(x)}{2},$$

and $\mathbb{C}$ is the elasticity tensor, which in terms of the Lame's parameters may be written as

$$\mathbb{C}A = 2\mu A + \lambda(\mathrm{tr}A)I,$$

for every symmetric matrix A, where I denotes the identity matrix and $\mathrm{tr}A$ denotes the trace of A. We assume $\mu > 0$ and $\lambda \geq 0$ so to have, for every symmetric tensor A,

$$\mathbb{C}A\cdot A \geq \mu|A|^2. \tag{2}$$

As customary in dimension reduction problems we pass to a domain independent of ε. Let $\Omega := \Omega_1$ and let $p_\varepsilon : \Omega \to \Omega_\varepsilon$ be defined by $p_\varepsilon(x) = p_\varepsilon(x_1, x_2, x_3) = (\varepsilon x_1, \varepsilon x_2, x_3)$. Following Ciarlet and Destuynder [2], we also change the name of the unknowns by setting

$$(u_1, u_2, u_3) := (\varepsilon\tilde{u}_1, \varepsilon\tilde{u}_2, \tilde{u}_3)\circ p_\varepsilon : \Omega \to \mathbb{R}^3.$$

We then have

$$D\tilde{u}\circ p_\varepsilon = \begin{pmatrix} \frac{1}{\varepsilon^2}D_\alpha u_\beta & \frac{1}{\varepsilon}D_3 u_\beta \\ \frac{1}{\varepsilon}D_\alpha u_3 & D_3 u_3 \end{pmatrix} =: H^\varepsilon u$$

where D_i denotes the partial derivatives with respect to x_i. We shall denote by

$$E^\varepsilon u := \frac{H^\varepsilon u + H^\varepsilon u^\mathsf{T}}{2}, \quad W^\varepsilon u := \frac{H^\varepsilon u - H^\varepsilon u^\mathsf{T}}{2},$$

the symmetric and the skew-symmetric part of $H^\varepsilon u$.

We consider loads of the form

$$b_1^\varepsilon \circ p_\varepsilon(x) = \varepsilon b_1(x) - \frac{m(x_3)}{I_0}x_2, \quad b_2^\varepsilon \circ p_\varepsilon(x) = \varepsilon b_2(x) + \frac{m(x_3)}{I_0}x_1,$$
$$b_3^\varepsilon \circ p_\varepsilon(x) = b_3(x),$$

with $b \in L^2(\Omega;\mathbb{R}^3)$, $m \in L^2(0,\ell)$, and where $I_0 := \int_\omega (x_1^2 + x_2^2)\, dx_1 dx_2$ denotes the polar moment of inertia of the section ω. With this notation the energy rewrites as

$$\begin{aligned} F_\varepsilon(u) \quad &:= \quad \frac{1}{\varepsilon^2}\mathcal{F}_\varepsilon\big((\frac{u_1}{\varepsilon}, \frac{u_2}{\varepsilon}, u_3) \circ p_\varepsilon^{-1}\big) \\ &= \quad I_\varepsilon(u) - \int_\Omega b \cdot u\, dx - \int_0^\ell m\, \vartheta^\varepsilon(u)\, dx_3, \end{aligned}$$

where

$$I_\varepsilon(u) := \frac{1}{2}\int_\Omega \mathbb{C}E^\varepsilon u \cdot E^\varepsilon u\, dx, \tag{3}$$

and

$$\vartheta^\varepsilon(u)(x_3) := \frac{1}{\varepsilon}\frac{1}{I_0}\int_\omega (x_1 u_2(x_1,x_2,u_3) - x_2 u_1(x_1,x_2,x_3))\, dx_1 dx_2. \tag{4}$$

The energy F_ε should be minimized over $H^1_{dn}(\Omega;\mathbb{R}^3) := H^1_{dn}(\Omega_1;\mathbb{R}^3)$.

3 Convergence of displacements

Throughout the section we consider a sequence $\{u^\varepsilon\} \subset H^1_{dn}(\Omega;\mathbb{R}^3)$ such that

$$\sup_{\varepsilon\in(0,1]} \|E^\varepsilon u^\varepsilon\|_{L^2(\Omega)} < +\infty. \tag{5}$$

Up to a subsequence, still denoted by ε, there exists an $E \in L^2(\Omega;\mathbb{R}^{3\times3})$ such that

$$E^\varepsilon u^\varepsilon \rightharpoonup E \text{ in } L^2(\Omega;\mathbb{R}^{3\times3}).$$

We start by studying the convergence of the displacements.

Theorem 3.1 *There exist a subsequence of* $\{u^\varepsilon\}$*, not relabeled, and a function*

$$u^0 \in H_{BN}(\Omega;\mathbb{R}^3) := \{v \in H^1_{dn}(\Omega;\mathbb{R}^3) \,:\, (Ev)_{i\alpha} = 0,\ i = 1,2,3,\ \alpha = 1,2\},$$

such that

$$u^\varepsilon \rightharpoonup u^0 \text{ in } H^1(\Omega;\mathbb{R}^3). \tag{6}$$

Moreover

$$E_{33} = D_3 u^0_3.$$

Proof. Since $|E^\varepsilon u^\varepsilon| \geq |Eu^\varepsilon|$ it follows, from (5), that Eu^ε is uniformly bounded in $L^2(\Omega;\mathbb{R}^{3\times 3})$ and, by Korn's inequality, that u^ε is uniformly bounded in $H^1(\Omega;\mathbb{R}^3)$. Hence, there exist a $u^0 \in H^1_{dn}(\Omega;\mathbb{R}^3)$ and a subsequence such that $u^\varepsilon \rightharpoonup u^0$ in $H^1(\Omega;\mathbb{R}^3)$. From the definition of E^ε we have that $|(E^\varepsilon u^\varepsilon)_{i\alpha}| \geq \frac{1}{\varepsilon}|(Eu^\varepsilon)_{i\alpha}|$, thus, using (5) we deduce that $C\varepsilon \geq \|(Eu^\varepsilon)_{i\alpha}\|_{L^2(\Omega)}$ and consequently $(Eu^0)_{i\alpha} = 0$. Hence $u^0 \in H_{BN}(\Omega;\mathbb{R}^3)$. The last part of the statement follows by noticing that $(E^\varepsilon u^\varepsilon)_{33} = D_3 u^\varepsilon_3$. □

The set $H_{BN}(\Omega;\mathbb{R}^3)$ is the space of Bernoulli-Navier displacements on Ω, and it can be characterized also as (see, for instance, Le Dret [9])

$$\begin{aligned} H_{BN}(\Omega;\mathbb{R}^3) = \{v \in H^1_{dn}(\Omega;\mathbb{R}^3) : \exists\, \xi_\alpha \in H^2_{dn}(0,\ell), \exists\, \xi_3 \in H^1_{dn}(0,\ell) \\ \text{such that } v_\alpha(x) = \xi_\alpha(x_3),\ v_3(x) = \xi_3(x_3) - x_\alpha \xi'_\alpha(x_3)\}. \end{aligned} \tag{7}$$

The characterization of the twist of the cross section will be much more involved. We start by stating a Korn's type inequality.

Theorem 3.2 *There exists a constant* $C_K > 0$ *such that*

$$\int_\Omega |\varepsilon H^\varepsilon v|^2 dx \leq C_K \int_\Omega |E^\varepsilon v|^2\, dx, \tag{8}$$

for every $v \in H^1_{dn}(\Omega;\mathbb{R}^3)$ *and every* $0 < \varepsilon \leq 1$.

Proof. This follows immediately by a result of Anzellotti et al. [1]. Indeed these authors have proved that there exists a constant $C_K > 0$ such that

$$\int_{\Omega_\varepsilon} |D\tilde{v}|^2 dx \leq \frac{C_K}{\varepsilon^2} \int_{\Omega_\varepsilon} |E\tilde{v}|^2\, dx, \tag{9}$$

for every $\tilde{v} \in H^1_{dn}(\Omega_\varepsilon;\mathbb{R}^3)$ and every $0 < \varepsilon \leq 1$. Rescaling this inequality to Ω we deduce the claim. $\square$

In the next Lemma we characterize the limit of the sequence $\{\varepsilon H^\varepsilon u^\varepsilon\}$ and, in doing so, we introduce the twist angle, ϑ, of the cross-section.

Lemma 3.1 *Let u^0 be as in Theorem 3.1. There exist a subsequence of $\{u^\varepsilon\}$, not relabeled, and a function $\vartheta \in L^2(\Omega)$ such that*

$$\varepsilon H^\varepsilon u^\varepsilon \rightharpoonup \begin{pmatrix} 0 & -\vartheta & D_3 u_1^0 \\ \vartheta & 0 & D_3 u_2^0 \\ -D_3 u_1^0 & -D_3 u_2^0 & 0 \end{pmatrix} \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}). \tag{10}$$

Proof. From (5) and Theorem 3.2 we deduce that the sequence $\varepsilon H^\varepsilon u^\varepsilon$ is bounded in $L^2(\Omega;\mathbb{R}^{3\times3})$ so that, up to subsequences, it weakly converges in $L^2(\Omega;\mathbb{R}^{3\times3})$ to a matrix $H \in L^2(\Omega;\mathbb{R}^{3\times3})$. Since, from (5), $\varepsilon E^\varepsilon u^\varepsilon \to 0$ in $L^2(\Omega;\mathbb{R}^{3\times3})$, we have $\varepsilon W^\varepsilon u^\varepsilon \rightharpoonup H$ in $L^2(\Omega;\mathbb{R}^{3\times3})$. In particular, H is, almost everywhere, a skew-symmetric matrix. Since $\varepsilon(H^\varepsilon u^\varepsilon)_{\alpha3} = D_3 u^\varepsilon_\alpha$, we deduce that $(H)_{\alpha3} = D_3 u^0_\alpha$. We conclude the proof by denoting $(H)_{12} := -\vartheta$. $\square$

The next lemma shows that ϑ is a function of x_3 only that is more regular than a square integrable function. Hereafter we denote by

$$x^{\mathsf{R}} := (-x_2, x_1).$$

Lemma 3.2 *With the notation introduced in* (4) *and in Lemma 3.1, we have that ϑ does not depend on x_1 and x_2, and*

1. $\exists C > 0 : \|\vartheta^\varepsilon(u^\varepsilon)\|_{L^2(0,\ell)} \leq C\|E^\varepsilon u^\varepsilon\|_{L^2(\Omega)} \quad \forall \varepsilon \in (0,1]$;
2. $\vartheta^\varepsilon(u^\varepsilon) \rightharpoonup \vartheta$ *in* $L^2(0,\ell)$;
3. $\vartheta \in H^1_{dn}(0,\ell) := \{\psi \in H^1(0,\ell) : \psi(0) = 0\}$.

Proof. From Theorem 3.1 and the definition of $H^\varepsilon(u^\varepsilon)$ we have that

$$\frac{1}{\varepsilon}(D_\alpha u^\varepsilon_\beta) \rightharpoonup \begin{pmatrix} 0 & -\vartheta \\ \vartheta & 0 \end{pmatrix} \text{ in } L^2(\Omega;\mathbb{R}^{2\times2}). \tag{11}$$

Set

$$w^\varepsilon_\beta(x_3) := \frac{u^\varepsilon_\beta}{\varepsilon}(\cdot,\cdot,x_3) - \frac{1}{|\omega|}\int_\omega \frac{u^\varepsilon_\beta}{\varepsilon}(x_1,x_2,x_3)\,dx_1dx_2. \tag{12}$$

By Poincare's inequality and Lemma 3.2, we have

$$\|w^\varepsilon\|^2_{L^2(0,\ell;L^2(\omega))} \leq \sum_{\alpha,\beta=1}^{2} \|\frac{1}{\varepsilon} D_\alpha u^\varepsilon_\beta\|^2_{L^2(\Omega)} \leq C\|\varepsilon H^\varepsilon u^\varepsilon\|^2_{L^2(\Omega)} \leq C\|E^\varepsilon u^\varepsilon\|^2_{L^2(\Omega)}, \tag{13}$$

and hence w^ε_β is a bounded sequence in $L^2(0,\ell;H^1(\omega))$. Thus up to a subsequence, not relabeled, we have that

$$w^\varepsilon_\beta \rightharpoonup w_\beta \text{ in } L^2(0,\ell;H^1(\omega)), \tag{14}$$

for some $w_\beta \in L^2(0,\ell;H^1(\omega))$. Since $D_\alpha w^\varepsilon_\beta = 1/\varepsilon D_\alpha u^\varepsilon_\beta$ we have, by (11), that

$$(D_\alpha w_\beta) = \begin{pmatrix} D_1 w_1 & D_2 w_1 \\ D_1 w_2 & D_2 w_2 \end{pmatrix} = \begin{pmatrix} 0 & -\vartheta \\ \vartheta & 0 \end{pmatrix}.$$

From two of these equations we deduce that w_1, respectively w_2, does not depend on x_1, respectively on x_2, and from the remaining two equations we find that $\vartheta = \vartheta(x_3)$. Also, by integration, we find

$$w(x_3)(x_1,x_2) = a(x_3) + x^{\mathsf{R}}\vartheta(x_3), \tag{15}$$

where we denoted by $w = (w_1, w_2)$, and where $a \in L^2(0,\ell)$.

Since the axes have origin in the center of mass, see (1), we may rewrite $\vartheta^\varepsilon(u^\varepsilon)$, see (4), as

$$\vartheta^\varepsilon(u^\varepsilon) = \frac{1}{\varepsilon}\frac{1}{I_0}\int_\omega (x_1 u^\varepsilon_2 - x_2 u^\varepsilon_1)\, dx_1 dx_2 = \frac{1}{I_0}\int_\omega x^{\mathsf{R}} \cdot w^\varepsilon \, dx_1 dx_2,$$

and

$$\vartheta = \frac{1}{I_0}\int_\omega x^{\mathsf{R}} \cdot x^{\mathsf{R}}\vartheta \, dx_1 dx_2 = \frac{1}{I_0}\int_\omega x^{\mathsf{R}} \cdot w \, dx_1 dx_2.$$

Thus, from (13) we deduce item *1.*, and from (14) we deduce item *2.* of the Lemma.

We now prove item *3.* Let $\psi \in C^\infty_0(\omega)$ be such that

$$\int_\omega \psi \, dx_1 dx_2 = \frac{I_0}{2},$$

and set

$$\tilde{\vartheta}^\varepsilon := \frac{1}{I_0}\int_\omega \operatorname{curl}\, \psi \cdot w^\varepsilon \, dx_1 dx_2,$$

where

$$\operatorname{curl}\,\psi := (D_2\psi, -D_1\psi).$$

Then, from (11) we deduce that

$$\tilde{\vartheta}^\varepsilon \rightharpoonup \frac{1}{I_0}\int_\omega \operatorname{curl}\,\psi \cdot w\, dx_1dx_2 \text{ in } L^2(0,\ell).$$

But

$$\begin{aligned}
\int_\omega \operatorname{curl}\,\psi \cdot w\, dx_1dx_2 &= a\int_\omega \operatorname{curl}\,\psi\, dx_1dx_2 + \vartheta\int_\omega \operatorname{curl}\,\psi \cdot x^{\mathsf{R}}\, dx_1dx_2 \\
&= -\vartheta\int_\omega D_\beta\psi\, x_\beta\, dx_1dx_2 = \vartheta\int_\omega \psi D_\beta x_\beta\, dx_1dx_2 \\
&= 2\vartheta\int_\omega \psi\, dx_1dx_2 = \vartheta.
\end{aligned}$$

Thus $\tilde{\vartheta}^\varepsilon \rightharpoonup \vartheta$ in $L^2(0,\ell)$. Since

$$\int_\omega \operatorname{curl}\,\psi\cdot(D_1u_3^\varepsilon, D_2u_3^\varepsilon)\, dx_1dx_2 = \int_{\partial\omega} u_3^\varepsilon \operatorname{curl}\,\psi\cdot n\, ds - \int_\omega u_3^\varepsilon \operatorname{div}\operatorname{curl}\,\psi\, dx_1dx_2 = 0,$$

we have that

$$\begin{aligned}
D_3\tilde{\vartheta}^\varepsilon &= \frac{1}{I_0}\int_\omega \operatorname{curl}\,\psi \cdot D_3w^\varepsilon\, dx_1dx_2 \\
&= \frac{1}{I_0}\int_\omega \operatorname{curl}\,\psi \cdot \big(\frac{D_3(u_1^\varepsilon, u_2^\varepsilon)}{\varepsilon} - \frac{1}{|\omega|}\int_\omega \frac{D_3(u_1^\varepsilon, u_2^\varepsilon)}{\varepsilon}\, dx_1dx_2\big)\, dx_1dx_2 \\
&= \frac{1}{I_0}\int_\omega \operatorname{curl}\,\psi \cdot \frac{D_3(u_1^\varepsilon, u_2^\varepsilon)}{\varepsilon}\, dx_1dx_2 \\
&= \frac{1}{I_0}\int_\omega \operatorname{curl}\,\psi \cdot \big(\frac{D_3(u_1^\varepsilon, u_2^\varepsilon)}{\varepsilon} + \frac{(D_1u_3^\varepsilon, D_2u_3^\varepsilon)}{\varepsilon}\big)\, dx_1dx_2 \\
&= \frac{2}{I_0}\int_\omega (\operatorname{curl}\,\psi)_\alpha (E^\varepsilon u^\varepsilon)_{\alpha 3}\, dx_1dx_2,
\end{aligned}$$

and hence $\tilde{\vartheta}^\varepsilon$ is bounded in $H^1(0,\ell)$. Thus $\tilde{\vartheta}^\varepsilon \in H^1_{dn}(0,\ell)$ and $\tilde{\vartheta}^\varepsilon \rightharpoonup \vartheta$ in $H^1(0,\ell)$. This convergence implies item *3.* of the Lemma. □

We now establish the relation between the twist angle ϑ and the limit strain E.

Theorem 3.3 *There exists a function* $z \in H^{-1}(0,\ell;L^2(\omega)) \subset \mathcal{D}'(0.\ell);L^2(\omega))$ *such that*

$$D_\alpha z = 2E_{\alpha 3} - (x^R)_\alpha D_3\vartheta \quad \textit{for } \alpha = 1,2,$$

and $\langle z,\varphi\rangle$ *has null average on* ω *for every* $\varphi \in \mathcal{D}(0,\ell)$.

Proof. Throughout the proof we shall denote the average on ω by

$$[\cdot] := \frac{1}{|\omega|}\int_\omega \cdot \, dx_1 dx_2,$$

and we shall use the notation and some of the results contained in the proof of Lemma 3.2. With the notation of the average just introduced we can rewrite, see (12), $w^\varepsilon_\beta = u^\varepsilon_\beta/\varepsilon - [u^\varepsilon_\beta/\varepsilon]$. Define

$$p^\varepsilon := \frac{1}{\varepsilon}\big(u^\varepsilon_3 - [u^\varepsilon_3] + D_3[u^\varepsilon_\beta]x_\beta\big),$$

and note that

$$D_\alpha p^\varepsilon = 2(E^\varepsilon u^\varepsilon)_{\alpha 3} - D_3 w^\varepsilon_\alpha. \tag{16}$$

Let $\psi \in C^\infty(0,\ell)$ be such that $0 \le \psi \le 1$, $\psi = 0$ on $(0,\ell/3)$, and $\psi = 1$ on $(2\ell/3,\ell)$, and let

$$p^\varepsilon_\psi(x_1,x_2,x_3) := \int_0^{x_3} p^\varepsilon(x_1,x_2,s)\psi(s)\,ds.$$

Since

$$\begin{aligned} D_\alpha p^\varepsilon_\psi(x_1,x_2,x_3) = &\int_0^{x_3} D_\alpha p^\varepsilon(x_1,x_2,s)\psi(s)\,ds = -w^\varepsilon_\alpha(x_1,x_2,x_3)\psi(x_3) \\ &+ \int_0^{x_3} 2(E^\varepsilon u^\varepsilon)_{\alpha 3}(x_1,x_2,s)\psi(s) + w^\varepsilon_\alpha(x_1,x_2,s)D_3\psi(s)\,ds, \end{aligned}$$

we have that

$$\|D_\alpha p^\varepsilon_\psi\|^2_{L^2(\Omega)} \le C_\psi(\|w^\varepsilon_\alpha\|^2_{L^2(\Omega)} + \|(E^\varepsilon u^\varepsilon)_{\alpha 3}\|^2_{L^2(\Omega)})$$

and so, by (14) and (5), $(D_\alpha p^\varepsilon_\psi)$ is a bounded sequence in $L^2(\Omega;\mathbb{R}^2)$. Since $[p^\varepsilon_\psi] = 0$, from Poincare's inequality we find that

$$\|p^\varepsilon_\psi\|^2_{L^2(\omega)} \le C\sum_{\alpha=1}^{2}\|D_\alpha p^\varepsilon_\psi\|^2_{L^2(\omega)},$$

and therefore (p^ε_ψ) is a bounded sequence in $L^2(0,\ell;H^1(\omega))$. From the definition of p^ε_ψ it follows that

$$\psi p^\varepsilon = D_3 p^\varepsilon_\psi$$

and hence

$$\begin{aligned}\|\psi p^\varepsilon\|_{H^{-1}(0,\ell;L^2(\omega))} &= \|D_3 p^\varepsilon_\psi\|_{H^{-1}(0,\ell;L^2(\omega))}\\ &= \sup_{\eta\in H^1_0(0,\ell;L^2(\omega))} \frac{\int_\Omega p^\varepsilon_\psi D_3\eta\,dx}{\|\eta\|_{H^1_0(0,\ell;L^2(\omega))}} \le \|p^\varepsilon_\psi\|_{L^2(0,\ell;L^2(\omega))}.\end{aligned}$$

Thus (ψp^ε) is a bounded sequence in $H^{-1}(0,\ell;L^2(\omega))$. Similarly we can show, by substituting in the previous argument $1-\psi$ to ψ, that $((1-\psi)p^\varepsilon)$ is a bounded sequence in $H^{-1}(0,\ell;L^2(\omega))$. Thus $p^\varepsilon = \psi p^\varepsilon + (1-\psi)p^\varepsilon$ is a bounded sequence in $H^{-1}(0,\ell;L^2(\omega))$ and hence there exists $p\in H^{-1}(0,\ell;L^2(\omega))$ such that, up to a subsequence,

$$p^\varepsilon \rightharpoonup p \text{ in } H^{-1}(0,\ell;L^2(\omega)).$$

Since $[p^\varepsilon_\psi]=0$, then $[\langle p,\varphi\rangle]=0$ for every $\varphi\in H^1_0(0;\ell)$.

Letting ε go to zero in (16) we find

$$D_\alpha p = 2E_{\alpha 3} - D_3 w_\alpha = 2E_{\alpha 3} - D_3(a + x^{\mathsf{R}}\vartheta)_\alpha,$$

where we have used (15). Keeping in mind that $a=a(x_3)$, we finally find

$$D_\alpha(p - x_\beta D_3 a_\beta) = 2E_{\alpha 3} - (x^{\mathsf{R}})_\alpha D_3\vartheta,$$

and the theorem is proven with $z = p - x_\beta D_3 a_\beta$. □

REMARKS.

1. For ω simply connected Theorem 3.3 can be proven in a much simpler way. We briefly outline the proof. Indeed, since

$$D_3(\varepsilon W^\varepsilon u^\varepsilon)_{12} = D_2(E^\varepsilon u^\varepsilon)_{13} - D_1(E^\varepsilon u^\varepsilon)_{23},$$

in the sense of distributions, recalling Lemma 3.1 and passing to the limit, we find

$$-D_3\vartheta = D_2E_{13} - D_1E_{23}.$$

The above identity can be equivalently written as

$$D_1(2E_{23} - x_1D_3\vartheta) - D_2(2E_{13} + x_2D_3\vartheta) = 0,$$

that is: the vector field with components $2E_{\alpha 3}-(x^{\mathsf{R}})_\alpha D_3\vartheta$ has curl equal to zero. Thus, for ω simply connected, see for instance Girault and Raviart [7], this implies the existence of a field z whose (bidimensional) gradient is the vector field $(2E_{\alpha 3}-(x^{\mathsf{R}})_\alpha D_3\vartheta)$.

2. Since $E\in L^2(\Omega;\mathbb{R}^{3\times 3})$ and $\vartheta\in H^1(0,\ell)$, from Theorem 3.3 we have that $z\in H^{-1}(0,\ell;H^1(\omega))$ and not just in $H^{-1}(0,\ell;L^2(\omega))$. Hereafter we shall denote by $H^{-1}(0,\ell;H^1_m(\omega))$ the set of functions $z\in H^{-1}(0,\ell;H^1(\omega))$ such that $\langle z,\varphi\rangle$ has null average on ω for every $\varphi\in C^\infty_0(0,\ell)$.

4 The limit energy

To state our main theorem we need to introduce some notation. The energy density for an isotropic material is

$$f(A)=\frac{1}{2}\mathbb{C}A\cdot A=\mu|A|^2+\frac{\lambda}{2}|\mathrm{tr}A|^2,$$

where λ and μ are the Lame's parameters, and $\mathbb{C}$ is the elasticity tensor which satisfies (2). The limit energy shall be defined by means of

$$f_0(\alpha_1,\alpha_2,\alpha_3):=\min\{f(A):A\in\mathrm{Sym},\ A_{i3}=\alpha_i,\ \text{for } i=1,2,3\},$$

that is

$$f_0(\alpha_1,\alpha_2,\alpha_3)=2\mu(\alpha_1^2+\alpha_2^2)+\frac{E}{2}\alpha_3^2, \tag{17}$$

where $E=\mu(2\mu+3\lambda)/(\mu+\lambda)$ is the Young modulus. The next lemma, which involves part of the energy density f_0 and the characterization of E_{13} and E_{23} given in Theorem 3.3, will be useful to deduce the limit energy.

Lemma 4.1

$$\begin{aligned}\inf\{2\mu\int_\Omega E_{13}^2+E_{23}^2\,dx\ :\ \exists\, z\in H^{-1}(0,\ell;H^1_m(\omega))\ s.t.\\ D_\alpha z=2E_{\alpha 3}-(x^R)_\alpha D_3\vartheta\ \textit{for}\ \alpha=1,2\}\\ =\frac{\mu}{2}\int_\omega|D\varphi+x^R|^2\,dx_1dx_2\int_0^\ell D_3\vartheta^2\,dx_3,\end{aligned}$$

where φ *is the* torsion function, *i.e.,*

$$\begin{cases} \triangle\varphi = 0 & \text{in } \omega, \\ D\varphi \cdot n = -x^{\mathsf{R}} \cdot n & \text{on } \partial\omega, \\ \int_\omega \varphi \, dx_1 dx_2 = 0. \end{cases} \tag{18}$$

Proof. Troughout the proof, D shall denote the bidimensional gradient. The infimum stated in the Lemma is equal to

$$\min_{z \in H^{-1}(0,\ell;H^1_m(\omega))} \frac{\mu}{2} \int_\Omega |Dz|^2 + 2Dz \cdot x^{\mathsf{R}} D_3\vartheta + |x^{\mathsf{R}} D_3\vartheta|^2 \, dx \tag{19}$$

where the minimum is actually achieved for $z = \hat\varphi \in L^2(0,\ell; H^1_m(\omega))$, as follows by an application of the direct method of the calculus of variations. Thus, $\hat\varphi(x_3) \in H^1_m(\omega)$ satisfies, for almost every $x_3 \in (0,\ell)$, the following problem

$$\begin{cases} \triangle\hat\varphi(x_3) = 0 \ \text{ in } \omega, \\ D\hat\varphi(x_3) \cdot n = -x^{\mathsf{R}} \cdot n D_3\vartheta(x_3) \ \text{ on } \partial\omega, \end{cases}$$

where n is the outer normal to $\partial\omega$. Thus

$$\hat\varphi = \varphi D_3\vartheta$$

where φ is the torsion function defined in problem (18). The end of the proof is simply achieved by computing the minimum value (in $z = \varphi D_3\vartheta$) of problem (19). $\square$

The quantity

$$\frac{\mu}{2} \int_\omega |D\varphi + x^{\mathsf{R}}|^2 \, dx_1 dx_2,$$

appearing in Lemma 4.1, is called the *torsional rigidity* of the beam. Usually it is not expressed in terms of the torsion function but by means of the so called Prandtl stress function that we are now going to introduce. This new function can be defined thanks to the following theorem whose proof can be found in Girault and Raviart [7].

Theorem 4.1 *A function* $v \in L^2(\omega;\mathbb{R}^2)$ *satisfies*

$$\begin{cases} \mathrm{div} v = 0 & \text{in } \omega, \\ \langle v \cdot n, 1\rangle_{H^{-1/2}(\gamma_h)\times H^{1/2}(\gamma_h)} = 0 & \text{for } h = 0,1,\dots,I, \end{cases}$$

if and only if there exists a function $g \in H^1(\omega)$ *such that*

$$v = \mathrm{curl} g := (D_2 g, -D_1 g).$$

The function $D\varphi+x^{\mathsf{R}}$, thanks to problem (18), satisfies the assumptions of Theorem 4.1 and therefore there exists a function $\psi \in H^1(\omega)$, called *Prandtl stress function*, such that

$$\mathrm{curl}\psi = D\varphi + x^{\mathsf{R}}. \tag{20}$$

Thus, the torsional rigidity can be simply written as

$$\frac{\mu}{2}\int_\omega |D\varphi + x^{\mathsf{R}}|^2\, dx_1 dx_2 = \frac{\mu}{2}\int_\omega |\mathrm{curl}\psi|^2\, dx_1 dx_2 = \frac{\mu}{2}\int_\omega |D\psi|^2\, dx_1 dx_2. \tag{21}$$

The Prandtl stress function can be computed directly without making use of the torsion function. Indeed from (20) we deduce that $\triangle\psi = -2$ in ω, and

$$\mathrm{curl}\,\psi\cdot n = 0 \text{ on each } \gamma_h. \tag{22}$$

Since $0 = \mathrm{curl}\psi \cdot n = D\psi \cdot t$, where $t = (-n_2, n_1)$ denotes the tangent unit vector to $\partial\omega$, we deduce that ψ is constant on each γ_h. Since ψ is defined up to a constant we may set $\psi = 0$ on γ_0 and $\psi = k_h$ for $h = 1, \dots, I$, where k_h are constants. Also, we have

$$0 = \int_{\gamma_h} D\varphi \cdot t\, ds = \int_{\gamma_h} (\mathrm{curl}\psi - x^{\mathsf{R}})\cdot t\, ds = \int_{\gamma_h} (-D\psi - x)\cdot n\, ds,$$

thus

$$\int_{\gamma_h} D\psi\cdot n\, ds = -\int_{\gamma_h} x\cdot n\, ds = \int_{\omega_h} \mathrm{div}x\, dx_1 dx_2 = 2A_h,$$

where A_h denotes the area of ω_h for $h = 1, \dots, I$. Therefore the Prandtl stress function satisfies

$$\begin{cases} \triangle\psi = -2 & \text{in } \omega, \\ \psi = 0 & \text{on } \gamma_0, \\ \psi = k_h & \text{on } \gamma_h \text{ for } h = 1, \dots, I, \\ \int_{\gamma_h} D\psi\cdot n\, ds = 2A_h, & \text{for } h = 1, \dots, I. \end{cases} \tag{23}$$

We now prove a technical lemma.

Lemma 4.2 *Let $\{u^\varepsilon\}$ be a sequence of functions in the space $H^1_{dn}(\Omega;\mathbb{R}^3)$. If $\sup_\varepsilon F_\varepsilon(u^\varepsilon) < +\infty$, then* (5) *holds for some constant $C > 0$.*

PROOF. By assumption there exists a constant $c > 0$ such that, for every ε we have

$$\begin{aligned} c \geq F_\varepsilon(u^\varepsilon) &\geq \frac{\mu}{2}\|E^\varepsilon u^\varepsilon\|^2_{L^2(\Omega)} - \|b\|_{L^2(\Omega)}\|u^\varepsilon\|_{L^2(\Omega)} - \|m\|_{L^2(0,\ell)}\|\vartheta^\varepsilon(u^\varepsilon)\|_{L^2(0,\ell)} \\ &\geq \frac{\mu}{4}\|E^\varepsilon u^\varepsilon\|^2_{L^2(\Omega)} + \frac{\mu}{4}\|Eu^\varepsilon\|^2_{L^2(\Omega)} - \eta(\|u^\varepsilon\|^2_{L^2(\Omega)} + \|\vartheta^\varepsilon(u^\varepsilon)\|^2_{L^2(0,\ell)}) - C_\eta \\ &\geq \frac{\mu}{4}\|E^\varepsilon u^\varepsilon\|^2_{L^2(\Omega)} + \frac{\mu}{4}\|Eu^\varepsilon\|^2_{L^2(\Omega)} - \eta\tilde{C}(\|Eu^\varepsilon\|^2_{L^2(\Omega)} + \|E^\varepsilon u^\varepsilon\|^2_{L^2(\Omega)}) - C_\eta \end{aligned}$$

where in the last inequality we have used Korn's inequality and item *1.* of Lemma 3.2. Since η is arbitrary, the claim follows by taking η small enough. $\square$

We finally state and prove our convergence result.

Theorem 4.2 *Let ψ be the Prandtl stress function defined above and let $F : H^1_{dn}(\Omega;\mathbb{R}^3) \times H^1_{dn}(0,\ell) \to \mathbb{R} \cup \{+\infty\}$ be defined by*

$$F(v,\vartheta) := \int_\Omega \frac{\mu}{2}|D\psi D_3\vartheta|^2 + \frac{E}{2}|D_3 v_3|^2\,dx - \int_\Omega b\cdot v\,dx - \int_0^\ell m\vartheta\,dx_3 \qquad (24)$$

if $v \in H_{BN}(\Omega;\mathbb{R}^3)$, and $+\infty$ otherwise. As $\varepsilon \to 0$, the sequence of functionals F_ε, defined in (3)-(4)*, Γ-converges to the functional F, in the following sense:*

1. *(liminf inequality) for every sequence of positive numbers ε_k converging to 0 and for every sequence $\{u^k\} \subset H^1_{dn}(\Omega;\mathbb{R}^3)$ such that*

$$u^k \rightharpoonup u^0 \text{ in } H^1(\Omega;\mathbb{R}^3), \qquad (\varepsilon_k W^{\varepsilon_k}u^k)_{12} \rightharpoonup -\vartheta \text{ in } L^2(\Omega),$$

we have

$$\liminf_{k\to+\infty} F_{\varepsilon_k}(u^k) \geq F(u^0,\vartheta);$$

2. *(recovery sequence) for every sequence of positive numbers ε_k converging to 0 and for every $(u^0,\vartheta) \in H^1_{dn}(\Omega;\mathbb{R}^3) \times H^1_{dn}(0,\ell)$ there exists a sequence $\{u^k\} \subset H^1_{dn}(\Omega;\mathbb{R}^3)$ such that*

$$u^k \rightharpoonup u^0 \text{ in } H^1(\Omega;\mathbb{R}^3), \qquad (\varepsilon_k W^{\varepsilon_k}u^k)_{12} \rightharpoonup -\vartheta \text{ in } L^2(\Omega),$$

and

$$\limsup_{k\to+\infty} F_{\varepsilon_k}(u^k) \leq F(u^0,\vartheta).$$

PROOF. Let us prove the liminf inequality. Without loss of generality we may suppose that

$$\liminf_{k\to+\infty} F_{\varepsilon_k}(u^k) = \lim_{k\to+\infty} F_{\varepsilon_k}(u^k) < +\infty.$$

Then Lemma 4.2 applies to the sequence $F_{\varepsilon_k}(u^k)$. Hence assumption (5) is fulfilled and the results of Section 3, namely Lemma 3.2 and Theorem 3.3, hold true.

Denoting the work done by loads, $L_\varepsilon := F_\varepsilon - I_\varepsilon$, where F_ε and I_ε are defined in Section 2, using Lemma 3.2 and the convergence assumptions on the sequence $\{u^k\}$, we can see that

$$L_{\varepsilon_k}(u^k) = \int_\Omega b\cdot u^k\,dx + \int_0^\ell m\vartheta^{\varepsilon_k}(u^k)\,dx_3 \to \int_\Omega b\cdot v\,dx + \int_0^\ell m\vartheta\,dx_3.$$

Thus we have only to prove that

$$\liminf_{k\to+\infty} I_{\varepsilon_k}(u^k) \geq \int_\Omega \frac{\mu}{2}|D\psi D_3\vartheta|^2 + \frac{E}{2}|D_3u_3^0|^2\,dx. \tag{25}$$

By definition of f_0 we get

$$\begin{aligned}
\liminf_{k\to+\infty} I_{\varepsilon_k}(u^k) &= \liminf_{k\to+\infty}\int_\Omega f(E^{\varepsilon_k}u^k)\,dx\\
&\geq \liminf_{k\to+\infty}\int_\Omega f_0((E^{\varepsilon_k}u^k)_{13},(E^{\varepsilon_k}u^k)_{23},(E^{\varepsilon_k}u^k)_{33})\,dx\\
&\geq \int_\Omega f_0(E_{13},E_{23},E_{33})\,dx\\
&= 2\mu\int_\Omega (E_{13}^2+E_{23}^2)\,dx + \frac{E}{2}\int_\Omega |E_{33}|^2\,dx,
\end{aligned}$$

where in the last inequality we have used the convexity of f_0, and where E_{i3} satisfy the properties stated in Theorem 3.1 and Theorem 3.3. Thus by Lemma 4.1, (21) and Theorem 3.3 we deduce (25).

Let us now find a recovery sequence. Let $F(u^0,\vartheta) < +\infty$, otherwise there is nothing to prove. Then $u^0 \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta \in H^1_{dn}(0,\ell)$.

We start by assuming that u^0 and ϑ are smooth and equal to zero near by $x_3 = 0$. By (7) there exists ξ smooth and equal to zero near by $x_3 = 0$ such that $u^0_\alpha(x) = \xi_\alpha(x_3)$, and $u^0_3(x) = \xi_3(x_3) - x_\alpha \xi'_\alpha(x_3)$. Let u^ε be the sequence defined by

$$
\begin{aligned}
u^\varepsilon_1 &= \xi_1 - \varepsilon x_2 \vartheta + \varepsilon^2 \frac{\nu}{2}\big(- x_2^2 \xi_1'' + x_1^2 \xi_1'' + 2x_1 x_2 \xi_2'' - 2x_1 \xi_3'\big) \\
u^\varepsilon_2 &= \xi_2 + \varepsilon x_1 \vartheta + \varepsilon^2 \frac{\nu}{2}\big(- x_1^2 \xi_2'' + x_2^2 \xi_2'' + 2x_1 x_2 \xi_1'' - 2x_2 \xi_3'\big) \\
u^\varepsilon_3 &= \xi_3 - x_1 \xi_1' - x_2 \xi_2' + \varepsilon \varphi D_3 \vartheta
\end{aligned}
\tag{26}
$$

where $\nu = \lambda/2(\lambda+\mu)$ is the Poisson's coefficient, and φ is the torsion function (with zero mean value). We have that u^ε is equal to zero in $x_3 = 0$ and

$$
\begin{aligned}
(\varepsilon W^\varepsilon u^\varepsilon)_{12} &= -\vartheta + O(\varepsilon) \\
\vartheta^\varepsilon(u^\varepsilon) &= \vartheta + O(\varepsilon), \\
E^\varepsilon u^\varepsilon &= Z(u^0_3, \vartheta) + O(\varepsilon),
\end{aligned}
$$

uniformly, where

$$
Z(u^0_3,\vartheta) := \begin{pmatrix} -\nu D_3 u^0_3, & 0 & (D_1\varphi - x_2) D_3\vartheta/2 \\ & -\nu D_3 u^0_3 & (D_2\varphi + x_1) D_3\vartheta/2 \\ \text{sym} & & D_3 u^0_3 \end{pmatrix}.
$$

By (20) we also have

$$
Z(u^0_3,\vartheta) = \begin{pmatrix} -\nu D_3 u^0_3, & 0 & D_2\psi D_3\vartheta/2 \\ & -\nu D_3 u^0_3 & -D_1\psi D_3\vartheta/2 \\ \text{sym} & & D_3 u^0_3 \end{pmatrix}, \tag{27}
$$

and a direct computation shows that

$$
f(Z(u^0_3,\vartheta)) = f_0(\frac{1}{2} D_2\psi D_3\vartheta, -\frac{1}{2} D_1\psi D_3\vartheta, D_3 u^0_3).
$$

Therefore,

$$
\begin{aligned}
I_\varepsilon(u^\varepsilon) &= \int_\Omega f(Z(u^0_3,\vartheta))\, dx + O(\varepsilon) \\
&= \int_\Omega f_0(\frac{1}{2} D_2\psi D_3\vartheta, -\frac{1}{2} D_1\psi D_3\vartheta, D_3 u^0_3)\, dx + O(\varepsilon) \\
&= \int_\Omega (\frac{\mu}{2} |D\psi D_3\vartheta|^2 + \frac{E}{2} |D_3 u^0_3|^2)\, dx + O(\varepsilon).
\end{aligned}
$$

Thus

$$F_\varepsilon(u^\varepsilon) = F(u^0, \vartheta) + O(\varepsilon),$$

which, together with the equalities above, implies that $\{u^{\varepsilon_k}\}$ is a recovery sequence. The proof of the general case, i.e., $u^0 \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta \in H^1_{dn}(0,\ell)$, is achieved by approximating u^0 in H^1 by smooth functions vanishing near by $x_3 = 0$ and concluding with a standard diagonal argument. $\square$

REMARK. If $v \in H_{BN}(\Omega;\mathbb{R}^3)$ then there exist $\xi_\alpha \in H^2_{dn}(0,\ell)$ and a $\xi_3 \in H^1_{dn}(0,\ell)$ such that $v_\alpha(x) = \xi_\alpha(x_3)$, $v_3(x) = \xi_3(x_3) - x_\alpha \xi'_\alpha(x_3)$. Then

$$\begin{aligned} F(v,\vartheta) = {} & \frac{1}{2}\int_0^\ell (EA{\xi'_3}^2 + EJ_2{\xi''_1}^2 + EJ_1{\xi''_2}^2 + \mu J_t{\vartheta'}^2)\,dx_3 \\ & - \int_0^\ell (f_1\xi_1 + f_2\xi_2 + f_3\xi_3 + c_1\xi'_1 + f_2\xi'_2 + m\vartheta)\,dx_3, \end{aligned}$$

where we simply denoted with a prime the derivative with respect to x_3 and where we set

$$\begin{aligned} A &:= \int_\omega dx_1dx_2, & J_1 &:= \int_\omega x_2^2\,dx_1dx_2, \\ J_t &:= \int_\omega |D\psi|^2\,dx_1dx_2, & J_2 &:= \int_\omega x_1^2\,dx_1dx_2, \end{aligned}$$

and

$$f_i := \int_\omega b_i\,dx_1dx_2, \quad c_\alpha := \int_\omega -x_\alpha b_3\,dx_1dx_2.$$

5 Error estimate

For every $\varepsilon \in (0,1]$ let $u^\varepsilon \in H^1_{dn}(\Omega;\mathbb{R}^3)$ be the minimizer of the three-dimensional problem, i.e.,

$$F_\varepsilon(u^\varepsilon) = \min_{v\in H^1_{dn}(\Omega;\mathbb{R}^3)} F_\varepsilon(v),$$

and let $(u,\vartheta) \in H_{BN}(\Omega;\mathbb{R}^3)\times H^1_{dn}(0,\ell)$ be the minimizer of the limit problem, that is

$$F(u,\vartheta) = \min_{(v,\check{\vartheta})\in H_{BN}(\Omega;\mathbb{R}^3)\times H^1_{dn}(0,\ell)} F(v,\check{\vartheta}).$$

From (7) there exist $\xi_\alpha \in H^2_{dn}(0,\ell)$ and $\xi_3 \in H^1_{dn}(0,\ell)$ such that

$$u_\alpha(x) = \xi_\alpha(x_3),\ u_3(x) = \xi_3(x_3) - x_\alpha \xi'_\alpha(x_3).$$

We make the following regularity assumption on the loads

$$(H) \qquad m \in L^2(0,\ell),\ b_\alpha \in L^2(\Omega),\ b_3 \in H^1(0,\ell; L^2(\omega)).$$

Under this assumption, using the minimality of (u,ϑ), we obtain that

$$\xi_\alpha \in H^2_{dn}(0,\ell) \cap H^4(0,\ell),\ \xi_3 \in H^1_{dn}(0,\ell) \cap H^3(0,\ell),\ \vartheta \in H^1_{dn}(0,\ell) \cap H^2(0,\ell). \tag{28}$$

To make a comparison between u^ε and the solution (u,ϑ) of the limit problem, we consider the sequence defined in (26) for (u,ϑ). In fact, in order to satisfy the Dirichlet boundary condition we correct such a sequence by means of a perturbation of order ε^2.

Let

$$\begin{aligned}
\gamma_1(x_1,x_2) &:= \frac{\nu}{2}\big(- x_2^2\xi''_1(0) + x_1^2\xi''_1(0) + 2x_1x_2\xi''_2(0) - 2x_1\xi'_3(0)\big),\\
\gamma_2(x_1,x_2) &:= \frac{\nu}{2}\big(- x_1^2\xi''_2(0) + x_2^2\xi''_2(0) + 2x_1x_2\xi''_1(0) - 2x_2\xi'_3(0)\big),\\
\gamma_3(x_1,x_2) &:= \varepsilon\varphi D_3\vartheta(0).
\end{aligned}$$

Let $\chi^\varepsilon : [0,\ell] \to [0,1]$ be the continuous piecewise affine function of x_3 defined by $\chi^\varepsilon(0) = 1$, $\chi^\varepsilon(\varepsilon) = \chi^\varepsilon(\ell) = 0$.

Let us denote by $\tilde{u}^\varepsilon$ the (recovery) sequence

$$\begin{aligned}
\tilde{u}^\varepsilon_1 &:= \xi_1 - \varepsilon x_2\vartheta + \varepsilon^2\frac{\nu}{2}\big(- x_2^2\xi''_1 + x_1^2\xi''_1 + 2x_1x_2\xi''_2 - 2x_1\xi'_3\big) - \varepsilon^2\gamma_1\chi^\varepsilon,\\
\tilde{u}^\varepsilon_2 &:= \xi_2 + \varepsilon x_1\vartheta + \varepsilon^2\frac{\nu}{2}\big(- x_1^2\xi''_2 + x_2^2\xi''_2 + 2x_1x_2\xi''_1 - 2x_2\xi'_3\big) - \varepsilon^2\gamma_2\chi^\varepsilon,\\
\tilde{u}^\varepsilon_3 &:= \xi_3 - x_1\xi'_1 - x_2\xi'_2 + \varepsilon\varphi D_3\vartheta - \varepsilon\gamma_3\chi^\varepsilon.
\end{aligned}$$

Theorem 5.1 *With the notation above and under assumption (H), there exists a constant C, independent of ε, such that*

$$\|E^\varepsilon(u^\varepsilon - \tilde{u}^\varepsilon)\|_{L^2(\Omega)} \le C\sqrt{\varepsilon}.$$

PROOF. Let

$$w^\varepsilon := u^\varepsilon - \tilde{u}^\varepsilon.$$

Since u^ε is a minimizer of F_ε, we have that

$$\begin{aligned} F^\varepsilon(\tilde{u}^\varepsilon) &\geq F^\varepsilon(u^\varepsilon) = F^\varepsilon(\tilde{u}^\varepsilon + w^\varepsilon) = \\ &= F^\varepsilon(\tilde{u}^\varepsilon) + \frac{1}{2}\int_\Omega \mathbb{C}E^\varepsilon w^\varepsilon \cdot E^\varepsilon w^\varepsilon \, dx - \int_\Omega b \cdot w^\varepsilon \, dx + \\ &\quad - \int_0^\ell m\, \vartheta^\varepsilon(w^\varepsilon)\, dx_3 + \int_\Omega \mathbb{C}E^\varepsilon \tilde{u}^\varepsilon \cdot E^\varepsilon w^\varepsilon \, dx, \end{aligned}$$

from which we deduce that

$$\|E^\varepsilon(w^\varepsilon)\|^2_{L^2(\Omega)} \leq C\Big|\int_\Omega \mathbb{C}E^\varepsilon \tilde{u}^\varepsilon \cdot E^\varepsilon w^\varepsilon \, dx - \int_\Omega b \cdot w^\varepsilon \, dx - \int_0^\ell m\, \vartheta^\varepsilon(w^\varepsilon)\, dx_3\Big|. \tag{29}$$

In the sequel we estimate the right hand side of the inequality above. We start by computing $\mathbb{C}E^\varepsilon \tilde{u}^\varepsilon$. A direct computation shows that

$$E^\varepsilon \tilde{u}^\varepsilon = Z + R^\varepsilon, \tag{30}$$

where $Z := Z(u, \vartheta)$ is defined in (27) and R^ε is defined by

$$R^\varepsilon_{11} = -D_1\gamma_1\chi^\varepsilon, \quad R^\varepsilon_{22} = -D_2\gamma_2\chi^\varepsilon, \quad R^\varepsilon_{33} = -\varepsilon\gamma_3\chi^{\varepsilon\prime},$$

$$R^\varepsilon_{12} = -\gamma\frac{D_2\gamma_1 + D_1\gamma_2}{2}\chi^\varepsilon,$$

$$R^\varepsilon_{13} = \varepsilon\frac{\nu}{4}\big(-x_2^2\xi_1''' + x_1^2\xi_1''' + 2x_1x_2\xi_2''' - 2x_1\xi_3''\big) - \frac{1}{2}\big(\varepsilon\gamma_1\chi^{\varepsilon\prime} + D_1\gamma_3\chi^\varepsilon\big),$$

$$R^\varepsilon_{23} = \varepsilon\frac{\nu}{4}\big(-x_1^2\xi_2''' + x_2^2\xi_2''' + 2x_1x_2\xi_1''' - 2x_2\xi_3''\big) - \frac{1}{2}\big(\varepsilon\gamma_2\chi^{\varepsilon\prime} + D_2\gamma_3\chi^\varepsilon\big).$$

From the definition of χ^ε we deduce that

$$\|R^\varepsilon\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon}, \tag{31}$$

where the constant C depends on the following norms: $\|\gamma_i\|_{L^2(\omega)}$, $\|D_\beta\gamma_\alpha\|_{L^2(\omega)}$, $\|D_\alpha\gamma_3\|_{L^2(\omega)}$, $\|\xi_\alpha'''\|_{L^2(0,\ell)}$, and $\|\xi_3''\|_{L^2(0,\ell)}$.

Since

$$(\mathbb{C}Z)_{\alpha\beta} = 0, \ (\mathbb{C}Z)_{13} = \mu D_2\psi\, \vartheta', \ (\mathbb{C}Z)_{23} = -\mu D_1\psi\, \vartheta', \ (\mathbb{C}Z)_{33} = ED_3u_3,$$

we find

$$\begin{aligned}
\int_\Omega \mathbb{C}Z\cdot E^\varepsilon w^\varepsilon\,dx &= \int_\Omega \mu D_2\psi\,\vartheta'\frac{D_3w_1^\varepsilon+D_1w_3^\varepsilon}{\varepsilon}-\mu D_1\psi\,\vartheta'\frac{D_3w_2^\varepsilon+D_2w_3^\varepsilon}{\varepsilon}+\\
&\qquad\qquad +ED_3u_3\,D_3w_3^\varepsilon\,dx\\
&= \int_\Omega \frac{\mu\vartheta'}{\varepsilon}\operatorname{curl}\psi\cdot(D_3\bar w^\varepsilon+\bar D w_3^\varepsilon)+ED_3u_3\,D_3w_3^\varepsilon\,dx \qquad (32)\\
&= \int_\Omega \frac{\mu\vartheta'}{\varepsilon}\operatorname{curl}\psi\cdot D_3\bar w^\varepsilon+ED_3u_3D_3w_3^\varepsilon\,dx,
\end{aligned}$$

where $\bar w^\varepsilon := (w_1^\varepsilon, w_2^\varepsilon)$, $\bar D := (D_1, D_2)$, and where in the last equality we used the divergence theorem and the condition $\operatorname{curl}\psi\cdot n=0$ on $\partial\omega$, see (22).

Substituting (30) in (29) and using (32) we find

$$\|E^\varepsilon(w^\varepsilon)\|^2_{L^2(\Omega)}\le C(I^\varepsilon+II^\varepsilon+III^\varepsilon), \qquad (33)$$

where

$$\begin{aligned}
I^\varepsilon &= \Big|\int_\Omega \mathbb{C}R^\varepsilon\cdot E^\varepsilon w^\varepsilon\,dx\Big|,\\
II^\varepsilon &= \Big|\int_\Omega \frac{\mu\vartheta'}{\varepsilon}\operatorname{curl}\psi\cdot D_3\bar w^\varepsilon\,dx-\int_0^\ell m\,\vartheta^\varepsilon(w^\varepsilon)\,dx_3\Big|,\\
III^\varepsilon &= \Big|\int_\Omega ED_3u_3D_3w_3^\varepsilon-b\cdot w^\varepsilon\,dx\Big|.
\end{aligned}$$

From (31) we easily find that

$$I^\varepsilon\le C\|R^\varepsilon\|_{L^2(\Omega)}\|E^\varepsilon w^\varepsilon\|_{L^2(\Omega)}\le C\sqrt{\varepsilon}\|E^\varepsilon w^\varepsilon\|_{L^2(\Omega)}. \qquad (34)$$

Concerning the second term in (33) we have

$$\begin{aligned}
II^\varepsilon &= \Big|\int_\Omega \frac{\mu\vartheta'}{\varepsilon}\operatorname{curl}\psi\cdot D_3(\varepsilon\vartheta^\varepsilon(w^\varepsilon)\operatorname{curl}\psi)\,dx-\int_0^\ell m\,\vartheta^\varepsilon(w^\varepsilon)\,dx_3+\\
&\quad+\int_\Omega \frac{\mu\vartheta'}{\varepsilon}\operatorname{curl}\psi\cdot D_3(\bar w^\varepsilon-\varepsilon\vartheta^\varepsilon(w^\varepsilon)\operatorname{curl}\psi)\,dx\Big|\\
&= \Big|\int_\Omega \mu|D\psi|^2\vartheta'\,\vartheta^\varepsilon(w^\varepsilon)'\,dx-\int_0^\ell m\,\vartheta^\varepsilon(w^\varepsilon)\,dx_3+\\
&\quad+\int_\Omega \frac{\mu\vartheta'}{\varepsilon}\operatorname{curl}\psi\cdot D_3\Big[\bar w^\varepsilon-\Big(\fint_\omega \bar w^\varepsilon\,dx_1dx_2+\varepsilon x^{\mathsf R}\vartheta^\varepsilon(w^\varepsilon)\Big)\Big]\,dx+\\
&\quad+\int_\Omega \frac{\mu\vartheta'}{\varepsilon}\operatorname{curl}\psi\cdot D_3\Big[\fint_\omega \bar w^\varepsilon\,dx_1dx_2+\varepsilon(x^{\mathsf R}-\operatorname{curl}\psi)\vartheta^\varepsilon(w^\varepsilon)\Big]\,dx\Big|.
\end{aligned}$$

The first line of the identity above is equal to zero because (u, ϑ) is a minimizer of F and hence it satisfies the Euler-Lagrange equation

$$\int_\Omega \mu |D\psi|^2 \vartheta' \, \eta' \, dx - \int_0^\ell m \, \eta \, dx_3 = 0 \quad \text{for every } \eta \in H^1_{dn}(\Omega; \mathbb{R}). \tag{35}$$

The last line is also equal to zero since, by (23), we have

$$\int_\omega \operatorname{curl} \psi \, dx_1 dx_2 = -\int_{\partial\omega} \psi \, t \, ds = -\sum_{h=1}^{I} k_h \int_{\gamma_h} t \, ds = 0,$$

and, using (20) and the fact that $\operatorname{curl} \psi \cdot n = 0$ on $\partial\omega$, we deduce

$$\begin{aligned} \int_\omega \operatorname{curl} \psi \cdot (x^{\mathsf{R}} - \operatorname{curl} \psi) \, dx_1 dx_2 &= -\int_\omega \operatorname{curl} \psi \cdot D\varphi \, dx_1 dx_2 \\ &= -\int_{\partial\omega} \varphi \operatorname{curl} \psi \cdot n \, ds = 0. \end{aligned}$$

Thus, since $\bar{w}^\varepsilon(x_1, x_2, 0) = 0$ and $\vartheta^\varepsilon(w^\varepsilon)(0) = 0$, and since $\mu\vartheta'(\ell) = 0$ as a consequence of (35), we find

$$\begin{aligned} II^\varepsilon &= \Big| \int_\Omega \frac{\mu\vartheta'}{\varepsilon} \operatorname{curl} \psi \cdot D_3 \big[\bar{w}^\varepsilon - (\fint_\omega \bar{w}^\varepsilon \, dx_1 dx_2 + \varepsilon x^{\mathsf{R}} \vartheta^\varepsilon(w^\varepsilon)) \big] \, dx \Big| \\ &= \Big| \int_\Omega \frac{\mu\vartheta''}{\varepsilon} \operatorname{curl} \psi \cdot \big[\bar{w}^\varepsilon - (\fint_\omega \bar{w}^\varepsilon \, dx_1 dx_2 + \varepsilon x^{\mathsf{R}} \vartheta^\varepsilon(w^\varepsilon)) \big] \, dx \Big| \\ &\leq \frac{C}{\varepsilon} \|\vartheta''\|_{L^2(0,\ell)} \big\| \bar{w}^\varepsilon - (\fint_\omega \bar{w}^\varepsilon \, dx_1 dx_2 + \varepsilon x^{\mathsf{R}} \vartheta^\varepsilon(w^\varepsilon)) \big\|_{L^2(\Omega)}. \end{aligned}$$

Applying the bi-dimensional Korn inequality

$$\big\| \bar{w}^\varepsilon - (\fint_\omega \bar{w}^\varepsilon \, dx_1 dx_2 + \varepsilon x^{\mathsf{R}} \vartheta^\varepsilon(w^\varepsilon)) \big\|_{L^2(\omega)} \leq C \|(Ew^\varepsilon)_{\alpha\beta}\|_{L^2(\omega)},$$

which holds almost everywhere in $(0, \ell)$, and, integrating over $(0, \ell)$, we find

$$\big\| \bar{w}^\varepsilon - (\fint_\omega \bar{w}^\varepsilon \, dx_1 dx_2 + \varepsilon x^{\mathsf{R}} \vartheta^\varepsilon(w^\varepsilon)) \big\|_{L^2(\Omega)} \leq C \|(Ew^\varepsilon)_{\alpha\beta}\|_{L^2(\Omega)} \leq C\varepsilon^2 \|E^\varepsilon w^\varepsilon\|_{L^2(\Omega)}.$$

Thus,

$$II^\varepsilon \leq C\varepsilon \|\vartheta''\|_{L^2(0,\ell)} \|E^\varepsilon w^\varepsilon\|_{L^2(\Omega)}. \tag{36}$$

Finally, we estimate III^ε.

Since (u,ϑ) is a minimizer of F, we have

$$\int_\Omega ED_3u_3D_3z_3\,dx = \int_\Omega b\cdot z\,dx \quad \text{for every } z\in H_{BN}(\Omega;\mathbb{R}^3). \tag{37}$$

Thus, for any $z^\varepsilon\in H_{BN}(\Omega;\mathbb{R}^3)$, we may write

$$\begin{aligned} III^\varepsilon &= \Big|\int_\Omega ED_3u_3D_3(w_3^\varepsilon - z_3) - b\cdot(w^\varepsilon - z^\varepsilon)\,dx\Big| \\ &= \Big|\int_\Omega (ED_3D_3u_3+b_3)(w_3^\varepsilon - z_3^\varepsilon) - b_\alpha(w^\varepsilon - z^\varepsilon)_\alpha\,dx\Big|. \end{aligned}$$

where we have used here the fact that $D_3u_3(x_1,x_2,\ell)=0$ which follows from (37) . We now take for z^ε the projection of w^ε on the Bernoulli-Navier space, that is

$$z_\alpha^\varepsilon = \fint_\omega w_\alpha^\varepsilon\,dx_1dx_2, \quad z_3^\varepsilon = \fint_\omega w_3^\varepsilon\,dx_1dx_2 - x_\alpha z_\alpha^{\varepsilon\,\prime}.$$

With this choice, using Poincaré inequality, we have

$$\|w_\alpha^\varepsilon - z_\alpha^\varepsilon\|_{L^2(\omega)} \le C\|D_\beta w_\alpha^\varepsilon\|_{L^2(\omega)} \le C\varepsilon^2\|H^\varepsilon w^\varepsilon\|_{L^2(\omega)}$$

and, integrating over $(0,\ell)$ and using Korn inequality, we deduce

$$\|w_\alpha^\varepsilon - z_\alpha^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon^2\|H^\varepsilon w^\varepsilon\|_{L^2(\Omega)} \le C\varepsilon\|E^\varepsilon w^\varepsilon\|_{L^2(\Omega)}.$$

To estimate $w_3^\varepsilon - z_3^\varepsilon$ we shall use a partial Korn inequality, see Theorem 7.2 and Theorem 8.1 of [11], which implies that

$$\begin{aligned} \|w_3^\varepsilon - z_3^\varepsilon\|^2_{(H^1(0,\ell;L^2(\omega)))'} &\le C\big(\|(Ew^\varepsilon)_{\alpha\beta}\|^2_{L^2(\Omega)} + \|(Ew^\varepsilon)_{\alpha 3}\|^2_{L^2(\Omega)} + \\ &\qquad\qquad + \varepsilon^3\|(E^\varepsilon w^\varepsilon)\|^2_{L^2(\Omega)}\big) \\ &\le C(\varepsilon^4+\varepsilon^2+\varepsilon^3)\|(E^\varepsilon w^\varepsilon)\|^2_{L^2(\Omega)} \\ &\le C\varepsilon^2\|(E^\varepsilon w^\varepsilon)\|^2_{L^2(\Omega)}. \end{aligned}$$

Hence

$$III^\varepsilon \le C\varepsilon\big(\|ED_3D_3u_3+b_3\|_{H^1(0,\ell;L^2(\omega))} + \|b\|_{L^2(\Omega)}\big)\|(E^\varepsilon w^\varepsilon)\|_{L^2(\Omega)},$$

where $\|ED_3D_3u_3+b_3\|_{H^1(0,\ell;L^2(\omega))}$ is finite by (28) and assumption (H). From (33), (34), (36), and the last inequality we obtain the claim of the Theorem. $\square$

REMARK. The rate of convergence $\sqrt{\varepsilon}$ appearing in Theorem 5.1 comes to the estimate of I^ε, and it is due to a boundary layer of width ε in which the boundary condition are accomodated. For an unconstrained beam we would obtain, with the same proof, an estimate of order ε instead of $\sqrt{\varepsilon}$.

Corollary 5.1 *Under the same notation and assumptions of Theorem 5.1, there exists a constant C, independent of ε, such that*

$$\|u^\varepsilon - u\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon}.$$

PROOF. By Korn inequality, for any $\varepsilon \in (0,1]$, we have

$$\|u^\varepsilon - \tilde{u}^\varepsilon\|_{H^1(\Omega)} \leq K\|E(u^\varepsilon - \tilde{u}^\varepsilon)\|_{L^2(\Omega)} \leq K\|E^\varepsilon(u^\varepsilon - \tilde{u}^\varepsilon)\|_{L^2(\Omega)} \leq C\sqrt{\varepsilon},$$

and the claim follows from the definition of $\tilde{u}^\varepsilon$. □

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