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An in-Depth Investigation of Interval Temporal Logic Model Checking with Regular Expressions

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***Abstract*—In this paper, we systematically investigate the model checking problem for interval temporal logic (ITL), where interval labeling is defined by means of regular expressions. In the last years, ITL model checking has received an increasing attention as a viable alternative to the traditional (point-based) temporal logic model checking, which can be recovered as a special case. Most results have been obtained by imposing suitable restrictions on interval labeling, by either defining it in terms of the labeling of interval endpoints or by constraining a proposition letter to hold over an interval if and only if it holds over each component state (homogeneity assumption). A possible way of overcoming these limitations has been recently outlined by Lomuscio and Michaliszyn, who proposed to exploit regular expressions to define the behavior of proposition letters over intervals in terms of the component states. They proved the decidability of ITL model checking with regular expressions for some very restricted fragments of Halpern and Shoham's interval temporal logic (HS), extended with epistemic operators, giving some rough upper bounds to its computational complexity. In this paper, we first prove that model checking for full HS with regular expressions is decidable. Then, we show that the formulas of a large class of HS fragments, namely, all fragments featuring (a subset of) HS modalities for the Allen's relations meets, met-by, starts, and started-by, can be model checked in polynomial working space (model checking for all these fragments turns out to be PSPACE-complete).**

I. Introduction

Model checking (MC) is commonly recognized as one of the most effective techniques in automatic system verification [BK08]. Besides in formal verification, it has been successfully used also in databases (e.g., active databases, database-backed web applications, NoSQL databases) and artificial intelligence (e.g., planning, configuration systems, multi-agent systems) [GM13], [GT99], [LQR09]. MC allows one to automatically checks whether a model of a given system satisfies a desired property to ensure that it meets the expected behaviour. A good balancing of expressiveness and complexity in the choice of the computational model and the specification formalism is a key factor for the actual exploitation of MC. Systems are usually modeled as finite-state transition graphs (finite Kripke structures), while properties are commonly expressed by formulas of point-based temporal logics, such as LTL, CTL, and CTL* [Pnu77], [EH86].

Various improvements to the computational model and/or the specification language have been proposed in the literature. As for the former, we mention MC for pushdown systems (see, e.g., [EHRS00]), that feature an infinite state space, while for the latter we remind the extensions of LTL with promptness, that make it possible to bound the delay with which a liveness request is fulfilled (see, e.g., [KPV09]).

In this paper, we focus on MC with interval temporal logic (ITL) as the specification language. ITL allows one to deal with relevant temporal properties, such as actions with duration, accomplishments, and temporal aggregations, which are inherently "interval-based" and cannot be properly expressed by point-based temporal logics. In the last years, ITL MC has received an increasing attention as a viable alternative to the traditional (point-based) temporal logic MC [Mon16], which can be recovered as a special case [BMM+16b].

ITLs feature intervals, instead of points, as their primitive temporal entities [HS91], [Mos83], [Ven90], and they have been fruitfully applied in various areas of computer science, including formal verification, computational linguistics, planning, and multi-agent systems [Mos83], [Pra05], [LM13]. Among ITLs, the landmark is Halpern and Shoham's modal logic of time intervals HS [HS91]. It features one modality for each of the 13 ordering relations between pairs of intervals (the so-called Allen's relations [All83]), apart from equality. Its satisfiability problem is undecidable over all relevant classes of linear orders [HS91], and most of its fragments are undecidable as well [BDG+14], [MM14]. Some meaningful exceptions are the logic of temporal neighbourhood $\mathsf{A\overline{A}}$ and the logic of sub-intervals D [BGMS09], [BGMS10].

The MC problem for HS and its fragments consists in the verification of the correctness of the behaviour of a given system with respect to some relevant interval properties. To make it effective, we need to collect information about states into computation stretches: we interpret each finite computation path as an interval, and we define its labelling on the basis of the labelling of the states that compose it. Most results have been obtained by imposing suitable restrictions on interval labeling: either a proposition letter can be constrained to hold over an interval if and only if it holds over each component state (homogeneity assumption [Roe80]), or interval labeling can be defined in terms of the labeling of interval endpoints.

In [MMM+16], Molinari et al. deal with MC for full HS over finite Kripke structures, under the homogeneity assumption, according to a state-based semantics that allows branching in the past and in the future. They introduce the fundamental elements of the problem and prove its non-elementary decidability and **PSPACE**-hardness. Since then, the attention was also brought to the fragments of HS, which, similarly to what happens with satisfiability, are often computationally much better [MMP15a], [MMP15b], [BMM+16a], [BMM+16c], [MMM+16], [MMPS16]. The MC problem for some HS fragments, extended with epistemic operators, has been investigated by Lomuscio and Michaliszyn in [LM13], [LM14]. Their semantic assumptions differ from those of [MMM+16], making it difficult to compare the two approaches. Formulae are interpreted over the unwinding of the Kripke structure (computation-tree-based semantics, according to [BMM+16b]), and interval labeling takes into account only the endpoints of intervals. The decidability status of MC for full epistemic HS is still unknown. An account of existing results can be found in Section III.

In [LM16], Lomuscio and Michaliszyn propose to exploit regular expressions to define the labeling of proposition letters over intervals in terms of the component states. They prove the decidability of MC with regular expressions for some very restricted fragments of epistemic HS, giving some rough upper bounds to its computational complexity. In this paper, we prove that MC with regular expressions for full HS is decidable. Moreover, we show that formulas of a large class of HS fragments, namely, those featuring (a subset of) HS modalities for the Allen's relations *meets, met-by, starts*, and *started-by* ($\mathsf{A\overline{A}B\overline{B}}$), can be checked in polynomial working space (MC for all these fragments turns out to be **PSPACE**-complete).

Structure of the paper: In Section II, we introduce the logic HS and provide some background knowledge. In Section III, we summarize known complexity results about MC for HS and its fragments. In Section IV, we prove that MC for full HS with regular expressions is decidable and that its complexity, when restricted to system models, that is, if we assume the formula to be constant length, is **NLOGSPACE**. In Section V, we prove a small-model theorem for the fragment $\mathsf{A\overline{A}B\overline{B}}$ (and the symmetric fragment $\mathsf{A\overline{A}E\overline{E}}$), that is exploited in Sections VI and VII to devise a **PSPACE** MC algorithm for $\mathsf{A\overline{A}B\overline{B}}$ (and $\mathsf{A\overline{A}E\overline{E}}$). Finally, in Section VII, we also prove that MC for the purely propositional fragment of HS, denoted as Prop, is hard for **PSPACE**: this is enough to conclude that MC for any sub-fragment of $\mathsf{A\overline{A}B\overline{B}}$ or $\mathsf{A\overline{A}E\overline{E}}$ is complete for **PSPACE**.

II. Preliminaries

In this section, we provide notation and background knowledge, and we introduce the interval temporal logic HS.

Let $\mathbb{N}$ be the set of natural numbers. For all $i, j \in \mathbb{N}$, we denote by $[i, j]$, with $i \leq j$, the set of naturals h such that $i \leq h \leq j$. Let Σ be an alphabet and w be a non-empty finite word over Σ. We denote by $|w|$ the length of w. For all $1 \leq i \leq j \leq |w|$, $w(i)$ denotes the i-th letter of w (i is called a w-position), while $w(i, j)$ denotes the finite subword of w given by $w(i) \cdots w(j)$. Let $|w| = n$. We define $\mathrm{fst}(w) = w(1)$ and $\mathrm{lst}(w) = w(n)$. $\mathrm{Pref}(w) = \{w(1, i) \mid 1 \leq i \leq n-1\}$ and $\mathrm{Suff}(w) = \{w(i, n) \mid 2 \leq i \leq n\}$ are the sets of all proper prefixes and suffixes of w, respectively. For $i \in [1, n]$, w^i is a shorthand for $w(1, i)$. The concatenation of two words w and w' is denoted as usual by $w \cdot w'$. Moreover, if $\mathrm{lst}(w) = \mathrm{fst}(w')$, $w \star w'$ represents $w(1, n-1) \cdot w'$. Finally, ε is the empty string.

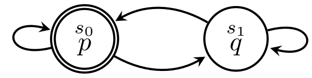


Fig. 1. The Kripke structure $\mathcal{K}_2$

For all $n, h \geq 0$, $\mathsf{Tower}(h, n)$ denotes a tower of exponentials of height h and argument n: $\mathsf{Tower}(0, n) = n$ and $\mathsf{Tower}(h+1, n) = 2^{\mathsf{Tower}(h,n)}$. Moreover, h-**EXPSPACE** denotes the class of languages decided by deterministic Turing machines bounded in space by functions of n in $O(\mathsf{Tower}(h, n^c))$, for some constant $c \geq 1$. Note that 0-**EXPSPACE** corresponds to **PSPACE**.

A. Kripke structures, regular expressions, and finite automata

Finite state systems are usually modelled as finite Kripke structures. Let $\mathcal{AP}$ be a finite set of proposition letters, which represent predicates over the states of the given system.

Definition 1 (Kripke structure)**.** *A* Kripke structure *over $\mathcal{AP}$ is a tuple $\mathcal{K} = (\mathcal{AP}, S, R, \mu, s_0)$, where S is a set of states, $R \subseteq S \times S$ is a left-total transition relation, $\mu : S \mapsto 2^{\mathcal{AP}}$ is a total labelling function assigning to each state s the set of proposition letters that hold over it, and $s_0 \in S$ is the initial state. For $s \in S$, the set $R(s)$ of successors of s is the non-empty set of states s' such that $(s, s') \in R$. We say that $\mathcal{K}$ is finite if S is finite.*

Fig. 1 depicts the finite Kripke structure $\mathcal{K}_2 = (\{p, q\}, \{s_0, s_1\}, R, \mu, s_0)$, where $R = \{(s_i, s_j) \mid i, j = 0, 1\}$, $\mu(s_0) = \{p\}$, $\mu(s_1) = \{q\}$. The initial state s_0 is marked by a double circle.

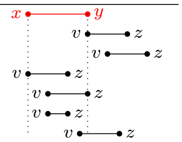
Let $\mathcal{K} = (\mathcal{AP}, S, R, \mu, s_0)$ be a Kripke structure. A *track* (or finite path) of $\mathcal{K}$ is a non-empty finite word ρ over S such that $(\rho(i), \rho(i+1)) \in R$ for all $i \in [1, |\rho| - 1]$. A track is *initial* if it starts from the initial state of $\mathcal{K}$. $\mathrm{Trk}_{\mathcal{K}}$ denotes the *infinite* set of tracks of $\mathcal{K}$. A track ρ induces the finite word $\mu(\rho)$ over $2^{\mathcal{AP}}$ given by $\mu(\rho(1)) \cdots \mu(\rho(n))$ where $n = |\rho|$. We call $\mu(\rho)$ the *trace* induced by ρ.

Let us introduce now the class of regular expressions over finite words. Since we are interested in expressing requirements over the traces induced by the tracks of Kripke structures, here we consider *proposition-based* regular expressions (RE), where the atomic formulas are propositional formulas over $\mathcal{AP}$ instead of letters over an alphabet. Formally, the set of RE r over $\mathcal{AP}$ is defined as follows:

$$r ::= \varepsilon \mid \phi \mid r \cup r \mid r \cdot r \mid r^*$$

TABLE I
ALLEN'S RELATIONS AND CORRESPONDING HS MODALITIES.

Allen relation	HS	Definition w.r.t. interval structures	Example
MEETS	$\langle \mathrm{A} \rangle$	$[x,y]\mathcal{R}_A[v,z] \iff y = v$	
BEFORE	$\langle \mathrm{L} \rangle$	$[x,y]\mathcal{R}_L[v,z] \iff y < v$	
STARTED-BY	$\langle \mathrm{B} \rangle$	$[x,y]\mathcal{R}_B[v,z] \iff x = v \wedge z < y$	
FINISHED-BY	$\langle \mathrm{E} \rangle$	$[x,y]\mathcal{R}_E[v,z] \iff y = z \wedge x < v$	
CONTAINS	$\langle \mathrm{D} \rangle$	$[x,y]\mathcal{R}_D[v,z] \iff x < v \wedge z < y$	
OVERLAPS	$\langle \mathrm{O} \rangle$	$[x,y]\mathcal{R}_O[v,z] \iff x < v < y < z$	



where ϕ is a propositional formula over $\mathcal{AP}$. The length $|r|$ of an RE r is the number of subexpressions of r. An RE r denotes a language $\mathcal{L}(r)$ of finite words over $2^{\mathcal{AP}}$ defined as:

- $\mathcal{L}(\varepsilon) = \{\varepsilon\}$ and $\mathcal{L}(\phi) = \{A \in 2^{\mathcal{AP}} \mid A \text{ satisfies } \phi\}$;
- $\mathcal{L}(r_1 \cup r_2) = \mathcal{L}(r_1) \cup \mathcal{L}(r_2)$, $\mathcal{L}(r_1 \cdot r_2) = \mathcal{L}(r_1) \cdot \mathcal{L}(r_2)$, and $\mathcal{L}(r^*) = (\mathcal{L}(r))^*$.

By well-known results, the class of RE over $\mathcal{AP}$ captures the class of regular languages of finite words over $2^{\mathcal{AP}}$.

As for finite automata, a non-deterministic finite automaton (NFA) is a tuple $\mathcal{A} = (\Sigma, Q, Q_0, \delta, F)$, where Σ is a finite alphabet, Q is a finite set of states, $Q_0 \subseteq Q$ is the set of initial states, $\delta : Q \times \Sigma \mapsto 2^Q$ is the transition function, and $F \subseteq Q$ is the set of accepting states. Given a finite word w over Σ, with $|w| = n$, and two states $q, q' \in Q$, a run (or computation) of $\mathcal{A}$ from q to q' over w is a finite sequence of states $q_1, \ldots, q_{n+1}$ such that $q_1 = q$, $q_{n+1} = q'$, and for all $i \in [1, n]$, $q_{i+1} \in \delta(q_i, w(i))$. The language $\mathcal{L}(\mathcal{A})$ accepted by $\mathcal{A}$ consists of the finite words w over Σ such that there is a run from some initial state to some accepting state over w.

A (complete) deterministic finite automaton (DFA) is an NFA $\mathcal{D} = (\Sigma, Q, Q_0, \delta, F)$ such that Q_0 is a singleton and for all $(q, c) \in Q \times \Sigma$, $\delta(q, c)$ is a singleton.

Remark 1. *By well-known results, given a* RE *r over $\mathcal{AP}$, one can construct, in a compositional way, an NFA $\mathcal{A}_r$ over $2^{\mathcal{AP}}$ such that $\mathcal{L}(\mathcal{A}_r) = \mathcal{L}(r)$, and whose number of states is at most $2|r|$. We call $\mathcal{A}_r$ the* canonical *NFA associated with r. Note that the number of edges of $\mathcal{A}_r$ may be exponential in $|\mathcal{AP}|$ (edges are labelled by assignments $A \in 2^{\mathcal{AP}}$ satisfying propositional formulas ϕ of r); however we can avoid storing edges, since they can be recovered in polynomial time from r.*

B. The interval temporal logic HS

An interval algebra to reason about intervals and their relative order was proposed by Allen in [All83], while a systematic logical study of interval representation and reasoning was done a few years later by Halpern and Shoham, who introduced the interval temporal logic HS featuring one modality for each Allen relation, but equality [HS91]. Table I depicts 6 of the 13 Allen's relations, together with the corresponding HS (existential) modalities. The other 7 relations are the 6 inverse relations (given a binary relation $\mathcal{R}$, the inverse relation $\overline{\mathcal{R}}$ is such that $b\overline{\mathcal{R}}a$ if and only if $a\mathcal{R}b$) and equality. Moreover, if $\langle X \rangle$ is the modality for $\mathcal{R}$, $\langle \overline{X} \rangle$ is the modality for $\overline{\mathcal{R}}$.

Given a finite set $\mathcal{P}_u$ of *uninterpreted interval properties*, the HS language over $\mathcal{P}_u$ consists of proposition letters from $\mathcal{P}_u$, the Boolean connectives $\neg$ and $\wedge$, the logical constants $\top$ and $\bot$, and a temporal modality for each of the (non trivial) Allen's relations, i.e., $\langle \mathrm{A} \rangle$, $\langle \mathrm{L} \rangle$, $\langle \mathrm{B} \rangle$, $\langle \mathrm{E} \rangle$, $\langle \mathrm{D} \rangle$, $\langle \mathrm{O} \rangle$, $\langle \overline{\mathrm{A}} \rangle$, $\langle \overline{\mathrm{L}} \rangle$, $\langle \overline{\mathrm{B}} \rangle$, $\langle \overline{\mathrm{E}} \rangle$, $\langle \overline{\mathrm{D}} \rangle$, and $\langle \overline{\mathrm{O}} \rangle$. HS formulas are defined by the grammar

$$\psi ::= p_u \mid \neg\psi \mid \psi \wedge \psi \mid \langle X \rangle \psi \mid \langle \overline{X} \rangle \psi,$$

where $p_u \in \mathcal{P}_u$ and $X \in \{A, L, B, E, D, O\}$. In the following, we will also use the standard connectives (disjunction $\vee$, implication $\rightarrow$, and double implication $\leftrightarrow$) as abbreviations. Furthermore, for any modality X, the dual universal modalities $[X]\psi$ and $[\overline{X}]\psi$ are defined as $\neg\langle X \rangle \neg\psi$ and $\neg\langle \overline{X} \rangle \neg\psi$, respectively. Given any subset of Allen's relations $\{X_1, \ldots, X_n\}$, we denote by $\mathsf{X_1 \cdots X_n}$ the HS fragment that features existential (and universal) modalities for $X_1, \ldots, X_n$ only.

W.l.o.g., we assume the *non-strict semantics of HS*, which admits intervals consisting of a single point[1]. Under such an assumption, all HS modalities can be expressed in terms of modalities $\langle \mathrm{B} \rangle, \langle \mathrm{E} \rangle, \langle \overline{\mathrm{B}} \rangle$, and $\langle \overline{\mathrm{E}} \rangle$ [Ven90]. For example, modality $\langle \mathrm{A} \rangle$ can be expressed in terms of $\langle \mathrm{E} \rangle$ and $\langle \overline{\mathrm{B}} \rangle$ as follows: $\langle \mathrm{A} \rangle \varphi := ([E]\bot \wedge (\varphi \vee \langle \overline{\mathrm{B}} \rangle \varphi)) \vee \langle \mathrm{E} \rangle ([E]\bot \wedge (\varphi \vee \langle \overline{\mathrm{B}} \rangle \varphi))$. HS can thus be viewed as a multi-modal logic with 4 primitive modalities. However, since later we will focus on the HS fragments $\mathsf{A\overline{A}E\overline{E}}$ and $\mathsf{A\overline{A}B\overline{B}}$—which respectively do not feature $\langle \mathrm{B} \rangle$, $\langle \overline{\mathrm{B}} \rangle$ and $\langle \mathrm{E} \rangle$, $\langle \overline{\mathrm{E}} \rangle$—we will sometimes explicitly add both $\langle \mathrm{A} \rangle$ and $\langle \overline{\mathrm{A}} \rangle$ to the considered set of HS modalities.

In [MMM$^+$16], the authors investigate the MC problem for finite Kripke structures $\mathcal{K}$ against HS formulas where the intervals correspond to the tracks of $\mathcal{K}$. The approach followed there is subject to two restrictions: (i) the set $\mathcal{P}_u$ of HS-proposition letters and the set $\mathcal{AP}$ of proposition letters for the Kripke structure coincide, and (ii) a proposition letter holds over an interval if and only if it holds over all its sub-intervals (homogeneity assumption). Here, we adopt a more general and expressive approach according to which an abstract interval proposition letter $p_u \in \mathcal{P}_u$ denotes a regular language of finite words over $2^{\mathcal{AP}}$. More specifically, each abstract interval proposition letter p_u is a (proposition-based) regular expression over $\mathcal{AP}$. Hence, hereafter, an HS formula over $\mathcal{AP}$ is an HS formula whose abstract interval proposition letters (or atomic formulas) are RE r over $\mathcal{AP}$.

Given a Kripke structure $\mathcal{K} = (\mathcal{AP}, S, R, \mu, s_0)$ over $\mathcal{AP}$, a track ρ of $\mathcal{K}$, and an HS formula φ over $\mathcal{AP}$, the satisfaction relation $\mathcal{K}, \rho \models \varphi$ is inductively defined as follows (we omit the clauses for the Boolean connectives, which are standard):

$$\begin{array}{ll}
\mathcal{K}, \rho \models r & \Leftrightarrow \mu(\rho) \in \mathcal{L}(r) \text{ for each RE } r \text{ over } \mathcal{AP}, \\
\mathcal{K}, \rho \models \langle \mathrm{B} \rangle \varphi & \Leftrightarrow \text{there exists } \rho' \in \mathrm{Pref}(\rho) \text{ s.t. } \mathcal{K}, \rho' \models \varphi, \\
\mathcal{K}, \rho \models \langle \mathrm{E} \rangle \varphi & \Leftrightarrow \text{there exists } \rho' \in \mathrm{Suff}(\rho) \text{ s.t. } \mathcal{K}, \rho' \models \varphi, \\
\mathcal{K}, \rho \models \langle \overline{\mathrm{B}} \rangle \varphi & \Leftrightarrow \mathcal{K}, \rho' \models \varphi \text{ for some track } \rho' \text{ s.t. } \rho \in \mathrm{Pref}(\rho'), \\
\mathcal{K}, \rho \models \langle \overline{\mathrm{E}} \rangle \varphi & \Leftrightarrow \mathcal{K}, \rho' \models \varphi \text{ for some track } \rho' \text{ s.t. } \rho \in \mathrm{Suff}(\rho').
\end{array}$$

$\mathcal{K}$ is a *model* of φ, denoted as $\mathcal{K} \models \varphi$, if for all *initial* tracks ρ of $\mathcal{K}$, it holds that $\mathcal{K}, \rho \models \varphi$. The HS *MC problem* is the problem of checking, for a finite Kripke structure $\mathcal{K}$ and an HS formula φ, whether or not $\mathcal{K} \models \varphi$. The problem is not trivially decidable since the set $\mathrm{Trk}_{\mathcal{K}}$ of tracks of $\mathcal{K}$ is infinite.

[1] All the results we prove in the paper hold for the strict semantics as well.

TABLE II
COMPLEXITY OF MC FOR HS AND ITS FRAGMENTS (†LOCAL MC).

	Homogeneity	Regular expressions	[LM13] – [LM16]
Full HS, BE	non-elem. **EXPSPACE**-hard	non-elem. **EXPSPACE**-hard	$\mathsf{BE}+KC^\dagger$: **PSPACE** $\mathsf{BE}^\dagger$: **P**
$\mathsf{A\overline{A}B\overline{B}\overline{E}}$, $\mathsf{A\overline{A}E\overline{B}\overline{E}}$	**EXPSPACE** **PSPACE**-hard	non-elem. **PSPACE**-hard	
$\mathsf{A\overline{A}\overline{B}\overline{E}}$	**PSPACE**-c.	non-elem. **PSPACE**-hard	
$\mathsf{A\overline{A}B\overline{B}}$, $\mathsf{B\overline{B}}$, $\mathsf{\overline{B}}$, $\mathsf{A\overline{A}E\overline{E}}$, $\mathsf{E\overline{E}}$, $\mathsf{\overline{E}}$	**PSPACE**-c.	**PSPACE**-c.	$\mathsf{A\overline{B}}+KC$: non-elem.
$\mathsf{A\overline{A}B}$, $\mathsf{A\overline{A}E}$, AB, $\mathsf{\overline{A}E}$	$\mathbf{P}^{\mathbf{NP}}$-c.	**PSPACE**-c.	
$\mathsf{A\overline{A}}$, $\mathsf{\overline{A}B}$, AE, A, $\mathsf{\overline{A}}$	$\mathbf{P}^{\mathbf{NP}[O(\log^2 n)]}$ $\mathbf{P}^{\mathbf{NP}[O(\log n)]}$-hard	**PSPACE**-c.	
Prop, B, E	**co-NP**-c.	**PSPACE**-c.	

III. THE GENERAL PICTURE

In this section, we give a short account of research on MC for HS and its fragments, in order to enlighten the original contributions of the present paper. Known and new results are summarized in Table II.

Let us consider first the MC problem for HS and its fragments under the homogeneity assumption. In [MMM$^+$16], Molinari et al. prove that MC for (full) HS is decidable. They provide an algorithm of non-elementary complexity, that, given a finite Kripke structure $\mathcal{K}$ and a bound k on the nesting depth of $\langle \mathrm{E} \rangle$ and $\langle \mathrm{B} \rangle$ modalities in the input HS formula, exploits a *finite* and satisfiability-equivalent representation for the infinite set $\mathrm{Trk}_{\mathcal{K}}$, that accounts for $\mathcal{K}$ and k. $\mathbf{EXPSPACE}$-hardness of BE, and thus of full HS, has been shown in [BMM$^+$16a]. An $\mathbf{EXPSPACE}$ MC algorithm for $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ and $\mathsf{A\overline{A}E\overline{B}\overline{E}}$ has been devised in [MMP15b]. For any track of $\mathcal{K}$, it finds a satisfiability-preserving track of bounded length (*track representative*). In this way, it only needs to check tracks with a bounded maximum length. $\mathbf{PSPACE}$-hardness of MC for $\mathsf{A\overline{A}B\overline{B}\overline{E}}$ and $\mathsf{A\overline{A}E\overline{B}\overline{E}}$ has been proved in [MMP15a]. Finally, it has been shown that formulas satisfying a constant bound on the nesting depth of $\langle \mathrm{B} \rangle$ (resp., $\langle \mathrm{E} \rangle$) can be checked in polynomial working space [MMP15b]. A number of well-behaved HS fragments, whose MC problem has a computational complexity markedly lower than that of full HS, have been identified in [MMP15a], [MMPS16], [BMM$^+$16a], [BMM$^+$16c], where the authors prove that MC is (i) $\mathbf{PSPACE}$-complete for $\mathsf{A\overline{A}\overline{B}\overline{E}}$, $\mathsf{A\overline{A}B\overline{B}}$, $\mathsf{A\overline{A}E\overline{E}}$, $\mathsf{\overline{B}}$, and $\mathsf{\overline{E}}$, (ii) in between $\mathbf{P}^{\mathbf{NP}[O(\log n)]}$ and $\mathbf{P}^{\mathbf{NP}[O(\log^2 n)]}$ for $\mathsf{A\overline{A}}$, A, $\mathsf{\overline{A}}$, $\mathsf{\overline{A}B}$, and AE, (iii) $\mathbf{P}^{\mathbf{NP}}$-complete for AB, $\mathsf{A\overline{A}B}$, $\mathsf{\overline{A}E}$, and $\mathsf{A\overline{A}E}$, and (iv) $\mathbf{co\text{-}NP}$-complete for B, E, and Prop.

It is interesting to compare such a complexity picture with the one emerging from the present paper. While relaxing the homogeneity assumption via regular expressions comes at no cost for $\mathsf{A\overline{A}B\overline{B}}$, $\mathsf{A\overline{A}E\overline{E}}$, $\mathsf{\overline{B}}$, $\mathsf{\overline{E}}$, $\mathsf{B\overline{B}}$, $\mathsf{E\overline{E}}$, $\ldots$ (they remain in $\mathbf{PSPACE}$), the complexity of the "lower" fragments B, E, $\mathsf{A\overline{A}}$, $\mathsf{A\overline{A}B}$, $\mathsf{A\overline{A}E}$, $\ldots$ increases to $\mathbf{PSPACE}$ (Prop will be proved to be $\mathbf{PSPACE}$-hard in Section VII, and its hardness immediately propagates to all the other HS fragments).

MC for some HS fragments, extended with the epistemic operators K and C, has been studied by Lomuscio and Michaliszyn, under the simplifying assumption that interval labeling is defined in terms of interval endpoints only. In [LM13], they focus on BED (BED is as expressive as BE since D can be defined by B and E), whose modalities allow one to respectively access prefixes, suffixes, and sub-intervals of the current interval. They consider a restricted form of MC (*local* MC), which verifies the given specification against a single (finite) initial computation interval, and prove that it is $\mathbf{PSPACE}$-complete. Moreover, they show that MC for the purely temporal fragment of BED is in $\mathbf{P}$. This is not surprising as BED modalities allow one to access only sub-intervals of the initial one, whose number is quadratic in the length (number of states) of the interval. In [LM14], they show that the picture drastically changes with HS fragments that allow one to access infinitely many tracks/intervals. In particular, they prove that MC for epistemic $\mathsf{A\overline{B}L}$ ($\mathsf{A\overline{B}L}$ is as expressive as $\mathsf{A\overline{B}}$ since L can be defined by A), whose modalities allow one to access intervals which are met by (respectively, extend to the right, follow) the current one, is decidable, by providing a non-elementary upper bound to the complexity of the problem. In [LM16], they propose an alternative definition of interval labeling, which associates each proposition letter with a regular expression over the set of states of the Kripke structure, that leads to a significant increase in expressiveness, as the labeling of an interval is no more determined by its endpoints, but it depends on the ordered sequence of states the interval consists of. Such a change does not cause any increase in computational complexity: (local) MC for BED is still in $\mathbf{PSPACE}$ and it is non-elementarily decidable for $\mathsf{A\overline{B}L}$. Nothing is said about MC for full HS, with or without epistemic operators.

In this paper, we define interval labeling via regular expressions in a way that, even though not identical to that of [LM16], can be shown to be equivalent. Moreover, it can be easily checked that the computation-tree-based and the state-based semantics behave exactly the same when restricted to HS fragments featuring present and future modalities only.[2] Hence, from the $\mathbf{PSPACE}$-completeness of $\mathsf{A\overline{A}B\overline{B}}$ proved in Section VII, it immediately follows that the sub-fragment $\mathsf{A\overline{B}L}$ with regular expressions, but without epistemic operators, can be checked in $\mathbf{PSPACE}$ (in fact, the non-elementary complexity of the MC algorithm for $\mathsf{A\overline{B}L}$ in [LM16] can be hardly ascribed to the addition of epistemic operators).

We conclude the section by underlining that the definitions of interval labeling given in [MMM$^+$16] and in [LM13], [LM14] are just special cases of the definition proposed here. To force homogeneity, it suffices to constrain all regular expressions in the formula to be of the form $p \cdot (p)^*$, for $p \in \mathcal{AP}$, while interval labeling based on endpoints can be captured by

[2] As shown in [BMM$^+$16b], this is not the case in general: the computation-tree-based semantics of [LM13], [LM14], [LM16] is subsumed by the state-based one of [MMM$^+$16] and follow-up papers.

regular expressions of the form $\bigcup_{(i,j)\in I}(q_i \cdot \top^* \cdot q_j)$, for some suitable $I \subseteq \{1,\dots,|S|\}^2$, where $q_i \in \mathcal{AP}$ is a proposition letter only labeling the state s_i of $\mathcal{K}$.

IV. MC FOR FULL HS

In this section, we provide an automata-theoretic solution to the MC problem for full HS. Given a finite Kripke structure $\mathcal{K}$ and an HS formula φ over $\mathcal{AP}$, we compositionally construct an NFA over the set of states of $\mathcal{K}$ accepting the set of tracks ρ such that $\mathcal{K},\rho \models \varphi$. The size of the resulting NFA is not elementary, but it is just linear in the size of the Kripke structure. In order to ensure that the non-elementary blow-up does not depend on the size of $\mathcal{K}$, we introduce a special subclass of NFA, that we call $\mathcal{K}$-NFA.

Fix a finite Kripke structure $\mathcal{K} = (\mathcal{AP}, S, R, \mu, s_0)$ over $\mathcal{AP}$.

Definition 2 ($\mathcal{K}$-NFA). *A $\mathcal{K}$-NFA is an NFA $\mathcal{A} = (S, Q, Q_0, \delta, F)$ over S satisfying the following requirements:*

- *the set Q of states is of the form $M \times S$: M is called the* main component *or the set of* main states*;*
- *$Q_0 \cap F = \emptyset$ (i.e., the empty word ε is not accepted);*
- *for all $(q,s) \in M \times S$ and $s' \in S$, $\delta((q,s),s') = \emptyset$, if $s' \neq s$, and $\delta((q,s),s) \subseteq M \times R(s)$.*

Note that a $\mathcal{K}$-NFA $\mathcal{A}$ accepts only tracks of $\mathcal{K}$. Moreover, for all words $\rho \in S^+$, if there is a run of $\mathcal{A}$ over ρ, then ρ is a track of $\mathcal{K}$. The following proposition holds.

Proposition 1. *Let $\mathcal{A}$ be an NFA over $2^{\mathcal{AP}}$ with n states. Then, one can construct in polynomial time a $\mathcal{K}$-NFA $\mathcal{A}_{\mathcal{K}}$ with at most $n+1$ main states accepting the set of tracks ρ such that $\mu(\rho) \in \mathcal{L}(\mathcal{A})$.*

Proof. Let $\mathcal{A} = (2^{\mathcal{AP}}, Q, Q_0, \delta, F)$. By using an additional state, we can assume that $\varepsilon \notin \mathcal{L}(\mathcal{A})$ (i.e., $Q_0 \cap F = \emptyset$). Then, $\mathcal{A}_{\mathcal{K}} = (S, Q \times S, Q_0 \times S, \delta', F \times S)$, where for all $(q,s) \in Q \times S$ and $s' \in S$, $\delta'((q,s),s') = \emptyset$, if $s' \neq s$, and $\delta'((q,s),s) = \delta(q,\mu(s)) \times R(s)$. Since $R(s) \neq \emptyset$ for all $s \in S$, correctness of the construction easily follows. ☐

We now extend in a natural way the semantics of the HS modalities $\langle \mathrm{B}\rangle$, $\langle \overline{\mathrm{B}}\rangle$, $\langle \mathrm{E}\rangle$, $\langle \overline{\mathrm{E}}\rangle$ over $\mathcal{K}$ to languages $\mathcal{L}$ of finite words over S. Given such a language $\mathcal{L}$ over S, let $\langle \mathrm{B}\rangle_{\mathcal{K}}(\mathcal{L})$, $\langle \mathrm{E}\rangle_{\mathcal{K}}(\mathcal{L})$, $\langle \overline{\mathrm{B}}\rangle_{\mathcal{K}}(\mathcal{L})$, $\langle \overline{\mathrm{E}}\rangle_{\mathcal{K}}(\mathcal{L})$ be the languages of tracks of $\mathcal{K}$ defined as follows:

- $\langle \mathrm{B}\rangle_{\mathcal{K}}(\mathcal{L}) = \{\rho \in \mathrm{Trk}_{\mathcal{K}} \mid \exists\, \rho' \in \mathcal{L} \cap S^+ \text{ and } \rho'' \in S^+ \text{ such that } \rho = \rho' \cdot \rho''\}$
- $\langle \overline{\mathrm{B}}\rangle_{\mathcal{K}}(\mathcal{L}) = \{\rho \in \mathrm{Trk}_{\mathcal{K}} \mid \exists\, \rho' \in S^+ \text{ s.t. } \rho \cdot \rho' \in \mathcal{L} \cap \mathrm{Trk}_{\mathcal{K}}\}$
- $\langle \mathrm{E}\rangle_{\mathcal{K}}(\mathcal{L}) = \{\rho \in \mathrm{Trk}_{\mathcal{K}} \mid \exists\, \rho'' \in \mathcal{L} \cap S^+ \text{ and } \rho' \in S^+ \text{ such that } \rho = \rho' \cdot \rho''\}$
- $\langle \overline{\mathrm{E}}\rangle_{\mathcal{K}}(\mathcal{L}) = \{\rho \in \mathrm{Trk}_{\mathcal{K}} \mid \exists\, \rho' \in S^+ \text{ s.t. } \rho' \cdot \rho \in \mathcal{L} \cap \mathrm{Trk}_{\mathcal{K}}\}$

The compositional translation of HS formulas into a $\mathcal{K}$-NFA is based on the following two propositions. First, we show that $\mathcal{K}$-NFAs are closed under the above language operations.

Proposition 2. *Given a $\mathcal{K}$-NFA $\mathcal{A}$ having n main states, one can construct in polynomial time $\mathcal{K}$-NFAs with $n+1$ main states accepting the languages $\langle \mathrm{B}\rangle_{\mathcal{K}}(\mathcal{L}(\mathcal{A}))$, $\langle \mathrm{E}\rangle_{\mathcal{K}}(\mathcal{L}(\mathcal{A}))$, $\langle \overline{\mathrm{B}}\rangle_{\mathcal{K}}(\mathcal{L}(\mathcal{A}))$, and $\langle \overline{\mathrm{E}}\rangle_{\mathcal{K}}(\mathcal{L}(\mathcal{A}))$, respectively.*

Proof. Let $\mathcal{A} = (S, M \times S, Q_0, \delta, F)$ be the given $\mathcal{K}$-NFA, where M is the set of main states. We now provide the construction for each of the considered language operations.

Construction for the language $\langle \mathrm{B}\rangle_{\mathcal{K}}(\mathcal{L}(\mathcal{A}))$: let us consider the NFA $\mathcal{A}_{\langle \mathrm{B}\rangle}$ over S given by $\mathcal{A}_{\langle \mathrm{B}\rangle} = (S, (M \cup \{q_{acc}\}) \times S, Q_0, \delta', \{q_{acc}\} \times S)$, where $q_{acc} \notin M$ is a fresh main state, and for all $(q,s) \in (M \cup \{q_{acc}\}) \times S$ and $s' \in S$, $\delta'((q,s),s') = \emptyset$, if $s' \neq s$, and $\delta'((q,s),s)$ is defined as follows:

$$\begin{cases} \delta((q,s),s) & \text{if } (q,s) \in (M \times S) \setminus F \\ \delta((q,s),s) \cup (\{q_{acc}\} \times R(s)) & \text{if } (q,s) \in F \\ \{q_{acc}\} \times R(s) & \text{if } q = q_{acc}. \end{cases}$$

Given an input word ρ, from an initial state (q_0,s) of $\mathcal{A}$, the automaton $\mathcal{A}_{\langle \mathrm{B}\rangle}$ simulates the behavior of $\mathcal{A}$ from (q_0,s) over ρ, but when $\mathcal{A}$ is in an accepting state (q_f,s) and the current input symbol is s, $\mathcal{A}_{\langle \mathrm{B}\rangle}$ can additionally choose to move to a state in $\{q_{acc}\} \times R(s)$, which is accepting for $\mathcal{A}_{\langle \mathrm{B}\rangle}$. From such states, $\mathcal{A}_{\langle \mathrm{B}\rangle}$ accepts iff the remaining portion of the input is a track of $\mathcal{K}$. Formally, by construction, since $\mathcal{A}$ is a $\mathcal{K}$-NFA, $\mathcal{A}_{\langle \mathrm{B}\rangle}$ is a $\mathcal{K}$-NFA as well. Moreover, a word ρ over S is accepted by $\mathcal{A}_{\langle \mathrm{B}\rangle}$ *iff* ρ is a track of $\mathcal{K}$ having some proper prefix ρ' in $\mathcal{L}(\mathcal{A})$ (note that $\rho' \neq \varepsilon$ since $\mathcal{A}$ is a $\mathcal{K}$-NFA). Hence, $\mathcal{L}(\mathcal{A}_{\langle \mathrm{B}\rangle}) = \langle \mathrm{B}\rangle_{\mathcal{K}}(\mathcal{L}(\mathcal{A}))$.

Construction for the language $\langle \mathrm{E}\rangle_{\mathcal{K}}(\mathcal{L}(\mathcal{A}))$: let us consider the NFA $\mathcal{A}_{\langle \mathrm{E}\rangle}$ over S given by $\mathcal{A}_{\langle \mathrm{E}\rangle} = (S, (M \cup \{q_0'\}) \times S, \{q_0'\} \times S, \delta', F)$, where $q_0' \notin M$ is a fresh main state and for all $(q,s) \in (M \cup \{q_0'\}) \times S$ and $s' \in S$, $\delta'((q,s),s') = \emptyset$, if $s' \neq s$, and $\delta((q,s),s)$ is defined as follows:

$$\begin{cases} \delta((q,s),s) & \text{if } q \neq q_0' \\ (\{q_0'\} \times R(s)) \cup \{(q_0,s') \in Q_0 \mid s' \in R(s)\} & \text{otherwise.} \end{cases}$$

Starting from an initial state (q_0',s), the automaton $\mathcal{A}_{\langle \mathrm{E}\rangle}$ either remains in a state whose main component is q_0', or moves to an initial state (q_0,s') of $\mathcal{A}$, ensuring at the same time that the portion of the input read so far is faithful to the evolution of $\mathcal{K}$. From the state (q_0,s'), $\mathcal{A}_{\langle \mathrm{E}\rangle}$ simulates the behavior of $\mathcal{A}$. Formally, since $\mathcal{A}$ is a $\mathcal{K}$-NFA, by construction it easily follows that $\mathcal{A}_{\langle \mathrm{E}\rangle}$ is a $\mathcal{K}$-NFA which accepts the set of tracks of $\mathcal{K}$ having a non-empty proper suffix in $\mathcal{L}(\mathcal{A})$. Hence, $\mathcal{L}(\mathcal{A}_{\langle \mathrm{E}\rangle}) = \langle \mathrm{E}\rangle_{\mathcal{K}}(\mathcal{L}(\mathcal{A}))$.

Construction for the language $\langle \overline{\mathrm{B}}\rangle_{\mathcal{K}}(\mathcal{L}(\mathcal{A}))$: let us consider the NFA $\mathcal{A}_{\langle \overline{\mathrm{B}}\rangle}$ over S given by $\mathcal{A}_{\langle \overline{\mathrm{B}}\rangle} = (S, (M \cup \{q_0'\}) \times S, \{q_0'\} \times S, \delta', F')$, where $q_0' \notin M$ is a fresh main state and δ' and F' are defined as follows:

- for all $(q,s) \in (M \cup \{q_0'\}) \times S$ and $s' \in S$, $\delta'((q,s),s') = \emptyset$, if $s' \neq s$, and $\delta'((q,s),s)$ is defined as follows:

$$\delta'((q,s),s) = \begin{cases} \bigcup_{(q_0,s) \in Q_0} \delta((q_0,s),s) & \text{if } q = q_0' \\ \delta((q,s),s) & \text{otherwise.} \end{cases}$$

- The set F' of accepting states is the set of states (q,s) of $\mathcal{A}$ such that there is a run of $\mathcal{A}$ from (q,s) to some state in F over some non-empty word.

Note that the set F' can be computed in time polynomial in the size of $\mathcal{A}$. Since $\mathcal{A}_{\langle \overline{\mathrm{B}}\rangle}$ essentially simulates $\mathcal{A}$ and $\{q_0'\} \times S$

and F' are disjoint, by construction it easily follows that $\mathcal{A}_{\langle\overline{\mathrm{B}}\rangle}$ is a $\mathcal{K}$-NFA. Moreover, $\mathcal{A}_{\langle\overline{\mathrm{B}}\rangle}$ accepts a word ρ iff ρ is a non-empty proper prefix of some word accepted by $\mathcal{A}$. Thus, since $\mathcal{A}$ is a $\mathcal{K}$-NFA, we obtain that $\mathcal{L}(\mathcal{A}_{\langle\overline{\mathrm{B}}\rangle}) = \langle\overline{\mathrm{B}}\rangle_{\mathcal{K}}(\mathcal{L}(\mathcal{A}))$.

Construction for the language $\langle\overline{\mathrm{E}}\rangle_{\mathcal{K}}(\mathcal{L}(\mathcal{A}))$*:* let us consider the NFA $\mathcal{A}_{\langle\overline{\mathrm{E}}\rangle}$ over S given by $\mathcal{A}_{\langle\overline{\mathrm{E}}\rangle} = (S, (M \cup \{q_{acc}\}) \times S, Q'_0, \delta', \{q_{acc}\} \times S)$, where $q_{acc} \notin M$ is a fresh main state, and Q'_0 and δ' are defined as follows:

- the set Q'_0 of initial states is the set of states (q, s) of $\mathcal{A}$ such that there is a run of $\mathcal{A}$ from some initial state to (q, s) over some non-empty word.
- For all $(q, s) \in (M \cup \{q_{acc}\}) \times S$ and $s' \in S$, $\delta'((q, s), s') = \emptyset$ if $s' \neq s$, and $\delta'((q, s), s)$ is as follows:

$$\begin{cases} \delta((q,s),s) \cup \bigcup\limits_{(q',s') \in F \cap \delta((q,s),s)} \{(q_{acc}, s')\} & \text{if } q \in M \\ \emptyset & \text{if } q = q_{acc}. \end{cases}$$

Note that the set Q'_0 can be computed in time polynomial in the size of $\mathcal{A}$. Since $\mathcal{A}_{\langle\overline{\mathrm{E}}\rangle}$ essentially simulates $\mathcal{A}$ and $\mathcal{A}$ is a $\mathcal{K}$-NFA, by construction, we easily obtain that $\mathcal{A}_{\langle\overline{\mathrm{E}}\rangle}$ is a $\mathcal{K}$-NFA which accepts the set of words over S which are non-empty proper suffixes of words in $\mathcal{L}(\mathcal{A})$. Thus, since $\mathcal{A}$ is a $\mathcal{K}$-NFA, we obtain that $\mathcal{L}(\mathcal{A}_{\langle\overline{\mathrm{E}}\rangle}) = \langle\overline{\mathrm{E}}\rangle_{\mathcal{K}}(\mathcal{L}(\mathcal{A}))$. This concludes the proof of Proposition 2. □

We now show that $\mathcal{K}$-NFAs are closed under Boolean operations.

Proposition 3. *Given two $\mathcal{K}$-NFAs $\mathcal{A}$ and $\mathcal{A}'$ with n and n' main states, respectively, the following holds:*

- Union: *one can construct in time $O(n + n')$ a $\mathcal{K}$-NFA with $n + n'$ main states accepting $\mathcal{L}(\mathcal{A}) \cup \mathcal{L}(\mathcal{A}')$.*
- Intersection: *one can construct in time $O(n \cdot n')$ a $\mathcal{K}$-NFA with $n \cdot n'$ main states accepting $\mathcal{L}(\mathcal{A}) \cap \mathcal{L}(\mathcal{A}')$.*
- Complementation: *one can construct in time $2^{O(n)}$ a $\mathcal{K}$-NFA with $2^{n+1} + 1$ main states accepting $\mathrm{Trk}_{\mathcal{K}} \setminus \mathcal{L}(\mathcal{A})$.*

Proof. Let $\mathcal{A} = (S, M \times S, Q_0, \delta, F)$ and $\mathcal{A}' = (S, M' \times S, Q'_0, \delta', F')$ be the given $\mathcal{K}$-NFAs. Without loss of generality, we assume that $M \cap M' = \emptyset$. We now provide the constructions for the considered language operations which are a natural generalization of the standard ones.

Union: let us consider the NFA $\mathcal{A}_\cup$ over S given by $\mathcal{A}_\cup = (S, (M \cup M') \times S, Q_0 \cup Q'_0, \delta'', F \cup F')$, where for all $(q, s) \in (M \cup M') \times S$ and $s' \in S$, $\delta''((q, s), s') = \emptyset$ if $s' \neq s$, and $\delta''((q, s), s)$ is defined as follows:

$$\delta''((q,s),s) = \begin{cases} \delta((q,s),s) & \text{if } q \in M \\ \delta'((q,s),s) & \text{if } q \in M'. \end{cases}$$

Correctness of the construction trivially follows.

Intersection: let us consider the NFA $\mathcal{A}_\cap$ over S given by $\mathcal{A}_\cap = (S, (M \times M') \times S, Q''_0, \delta'', F'')$, where Q''_0, δ'', and F'' are defined as follows:

- $Q''_0 = \{((q_0, q'_0), s) \mid (q_0, s) \in Q_0 \text{ and } (q'_0, s) \in Q'_0\}$.
- For all $((q, q'), s) \in (M \times M') \times S$ and all $s' \neq s$, $\delta''(((q, q'), s), s') = \emptyset$, and $\delta''(((q, q'), s), s) = \{((p, p'), s') \mid (p, s') \in \delta((q, s), s) \text{ and } (p', s') \in \delta'((q', s), s)\}$.
- $F'' = \{((q, q'), s) \mid (q, s) \in F \text{ and } (q', s) \in F'\}$.

Correctness of the construction easily follows.

Complementation: recall that $\mathcal{A} = (S, M \times S, Q_0, \delta, F)$. Let n be the number of main states of $\mathcal{A}$. First, we need a preliminary construction. Let us consider the NFA $\mathcal{A}'' = (S, (M \cup \{q_{acc}\}) \times S, Q_0, \delta'', \{q_{acc}\} \times S)$, where $q_{acc} \notin M$ is a fresh main state, and for all $(q, s) \in (M \cup \{q_{acc}\}) \times S$ and $s' \in s$, $\delta''((q, s), s') = \emptyset$ if $s' \neq s$, and $\delta''((q, s), s)$ is defined as follows:

$$\begin{cases} \delta((q,s),s) \cup (\{q_{acc}\} \times S) & \text{if } q \in M \text{ and } \delta((q,s),s) \cap F \neq \emptyset \\ \delta((q,s),s) & \text{if } q \in M \text{ and } \delta((q,s),s) \cap F = \emptyset \\ \emptyset & \text{if } q = q_{acc}. \end{cases}$$

Note that $\mathcal{A}''$ is *not* a $\mathcal{K}$-NFA. However, $\mathcal{L}(\mathcal{A}'') = \mathcal{L}(\mathcal{A})$.

Next we show that it is possible to construct in time $2^{O(n)}$ a *weak* $\mathcal{K}$-NFA $\mathcal{A}_c$ with 2^{n+1} main states accepting $(\mathrm{Trk}_{\mathcal{K}} \setminus \mathcal{L}(\mathcal{A}'')) \cup \{\varepsilon\}$, where a *weak* $\mathcal{K}$-NFA is a $\mathcal{K}$-NFA but the requirement that the empty word ε is not accepted is relaxed. Thus, since a weak $\mathcal{K}$-NFA can be easily converted into an equivalent $\mathcal{K}$-NFA by using an additional main state and $\mathcal{L}(\mathcal{A}'') = \mathcal{L}(\mathcal{A})$, the result follows. Let $M' = M \cup \{q_{acc}\}$. Then, the weak $\mathcal{K}$-NFA $\mathcal{A}_c$ is given by $\mathcal{A}_c = (S, 2^{M'} \times S, Q_{0,c}, \delta_c, F_c)$, where $Q_{0,c}$, δ_c, and F_c are defined as follows:

- $Q_{0,c} = \{(P, s) \in 2^M \times S \mid P = \{q \in M \mid (q, s) \in Q_0\}$;
- for all $(P, s) \in 2^{M'} \times S$ and $s' \in S$, $\delta_c((P, s), s') = \emptyset$ if $s' \neq s$, and $\delta_c((P, s), s)$ is given by

$$\bigcup_{s' \in R(s)} \{(\{q' \in M' \mid (q', s') \in \bigcup_{p \in P} \delta''(p, s)\}, s')\};$$

- $F_c = \{(P, s) \in 2^M \times S\}$.

By construction, $\mathcal{A}_c$ is a weak $\mathcal{K}$-NFA. Hence $\mathcal{A}_c$ does not accept words in $S^+ \setminus \mathrm{Trk}_{\mathcal{K}}$. Moreover, by construction $Q_{0,c} \subseteq F$. Hence $\varepsilon \in \mathcal{L}(\mathcal{A}_c)$. Let $\rho \in \mathrm{Trk}_{\mathcal{K}}$ with $|\rho| = k$. It remains to show that $\rho \in \mathcal{L}(\mathcal{A}'')$ iff $\rho \notin \mathcal{L}(\mathcal{A}_c)$.

First, let $\rho \in \mathcal{L}(\mathcal{A}'')$. We show that $\rho \notin \mathcal{L}(\mathcal{A}_c)$. We assume the contrary and derive a contradiction. Hence, there is a run of $\mathcal{A}_c$ over ρ of the form $(P_0, s_0), \ldots, (P_k, s_k)$ such that $(P_0, s_0) \in Q_{0,c}$ and $(P_k, s_k) \in F_c$. Hence $q_{acc} \notin P_k$. By construction, $P_0 = \{q \in M \mid (q, s_0) \in Q_0\}$, and for all $i \in [0, k-1]$, $s_i = \rho(i)$ and $P_{i+1} = \{p \in M' | (p, s_{i+1}) \in \delta''(q, s_i) \text{ for some } q \in P_i\}$. Since $\rho \in \mathcal{L}(\mathcal{A}'')$, there is $s \in S$, $(q_0, s_0) \in Q_0$ and an accepting run of $\mathcal{A}''$ over ρ of the form $(q_0, s_0), \ldots, (q_{k-1}, s_{k-1}), (q_k, s)$ where $q_k = q_{acc}$. By definition of the transition function of $\mathcal{A}''$, we can also assume that $s = s_k$. It follows that $q_i \in P_i$ for all $i \in [0, k]$, which is a contradiction since $q_{acc} \notin P_k$. Therefore, $\rho \notin \mathcal{L}(\mathcal{A}_c)$.

For the converse direction, let $\rho \notin \mathcal{L}(\mathcal{A}_c)$. We need to show that $\rho \in \mathcal{L}(\mathcal{A}'')$. By construction, there is some run of $\mathcal{A}_c$ over ρ starting from an initial state (recall that $R(s) \neq \emptyset$ for all $s \in S$). Moreover, each of such runs is of the form $(P_0, s_0), \ldots, (P_k, s_k)$ such that $P_0 = \{q \in M \mid (q, s_0) \in Q_0\}$, $q_{acc} \in P_k$, and for all $i \in [0, k-1]$, $s_i = \rho(i)$ and $P_{i+1} = \{p \in M' | (p, s_{i+1}) \in \delta(q, s_i) \text{ for some } q \in P_i\}$. It easily follows that there is an accepting run of $\mathcal{A}''$ over ρ from some initial state in $P_0 \times \{s_0\}$. Hence, the result follows.

This concludes the proof of Proposition 3. $\square$

Let φ be an HS formula. By using De Morgan's laws, we can trivially convert φ into an equivalent formula, called *negation form of* φ, using Boolean connectives and the modalities $\langle \mathrm{B} \rangle$, $\langle \overline{\mathrm{B}} \rangle$, $\langle \mathrm{E} \rangle$, $\langle \overline{\mathrm{E}} \rangle$, such that negation is applied only to the HS temporal modalities and to atomic formulas (i.e., regular expressions). For all $h \geq 1$, HS_h denotes the syntactical HS fragment consisting only of formulas φ such that the nesting depth of negation in the negation form of φ is at most h. Moreover $\neg\mathsf{HS}_h$ is the set of formulas φ such that $\neg\varphi \in \mathsf{HS}_h$.

Note that given an HS formula φ, checking whether $\mathcal{K} \not\models \varphi$ reduces to checking the existence of an initial track ρ of $\mathcal{K}$ such that $\mathcal{K}, \rho \models \neg\varphi$. Thus, since non-emptiness of NFAs is **NLOGSPACE**-complete, the automata constructions in Propositions 1–3 can be done on the fly, and **NLOGSPACE** = **co-NLOGSPACE**, by Propositions 1–3, we easily obtain the main result of this section.

Corollary 1. *There exists a constant c such that, given a finite Kripke structure $\mathcal{K}$ and an HS formula φ, one can construct a $\mathcal{K}$-NFA with $O(|\mathcal{K}| \cdot \mathsf{Tower}(h, |\varphi|^c))$ states accepting the set of tracks ρ such that $\mathcal{K}, \rho \models \varphi$, where h is the nesting depth of negation in the negation form of φ. Moreover, for each $h \geq 1$, the MC problem for $\neg\mathsf{HS}_h$ is in $(h-1)$-***EXPSPACE***. Additionally, for a fixed formula, the MC problem is* ***NLOGSPACE****-complete.*

V. Exponential Small-Model for $\mathsf{A\overline{A}B\overline{B}}$ and $\mathsf{A\overline{A}E\overline{E}}$

In this section we prove an exponential small-model property for the fragments $\mathsf{A\overline{A}B\overline{B}}$ and $\mathsf{A\overline{A}E\overline{E}}$, namely, if a track ρ of a finite Kripke structure $\mathcal{K}$ satisfies a formula φ of $\mathsf{A\overline{A}B\overline{B}}$ or $\mathsf{A\overline{A}E\overline{E}}$, then there exists a track π, whose length is exponential in the sizes of φ and $\mathcal{K}$, starting from and leading to the same states as ρ, that satisfies φ. In the following, we focus on $\mathsf{A\overline{A}B\overline{B}}$ (as the case for $\mathsf{A\overline{A}E\overline{E}}$ is completely symmetric).

Let $\mathcal{K} = (\mathcal{AP}, S, R, \mu, s_0)$ be a finite Kripke structure. We start by introducing the notion of *track induced* by a track ρ which is, intuitively, obtained by suitably contracting ρ concatenating some subtracks of ρ (provided that the resulting sequence is a track of $\mathcal{K}$ as well).

Definition 3. *Let $\rho \in \mathrm{Trk}_{\mathcal{K}}$ be a track with $|\rho| = n$. A* track induced by ρ *is a track $\pi \in \mathrm{Trk}_{\mathcal{K}}$ such that there exists an increasing sequence of ρ-positions $i_1 < \ldots < i_k$, with $i_1 = 1$, $i_k = n$, and $\pi = \rho(i_1) \cdots \rho(i_k)$. Moreover, we say that the π-position j and the ρ-position i_j are* corresponding.

Note that if π is induced by ρ, then $\mathrm{fst}(\pi) = \mathrm{fst}(\rho)$, $\mathrm{lst}(\pi) = \mathrm{lst}(\rho)$, and $|\pi| \leq |\rho|$ (in particular, $|\pi| = |\rho|$ iff $\pi = \rho$).

Given a DFA $\mathcal{D} = (\Sigma, Q, q_0, \delta, F)$ we denote by $\mathcal{D}(w)$ the state reached by the computation of $\mathcal{D}$ from q_0 over the word $w \in \Sigma^*$. Analogously, we denote by $\mathcal{D}_q(w)$ the state reached by the computation of $\mathcal{D}$ *from* $q \in Q$ over $w \in \Sigma^*$.

In the following we are interested in *well-formedness* of induced tracks with respect to a set of DFAs: a well formed track π induced by ρ preserves the states of the computations of the DFAs reached by reading prefixes of ρ and π bounded by corresponding positions (of ρ and π).

Definition 4. *Let $\mathcal{K} = (\mathcal{AP}, S, R, \mu, s_0)$ be a finite Kripke structure, $\rho \in \mathrm{Trk}_{\mathcal{K}}$ be a track, and $\mathcal{D}^s = (2^{\mathcal{AP}}, Q^s, q_0^s, \delta^s, F^s)$ with $s = 1, \ldots, k$, be DFAs. A track $\pi \in \mathrm{Trk}_{\mathcal{K}}$* induced by ρ *is $(q^1_{\ell_1}, \cdots, q^k_{\ell_k})$*-well-formed w.r.t. ρ, *with $q^s_{\ell_s} \in Q^s$ for all $s = 1, \ldots, k$, if and only if:*

- *for all π-positions j, with corresponding ρ-positions i_j, and for all $s = 1, \ldots, k$, $\mathcal{D}^s_{q^s_{\ell_s}}(\mu(\pi^j)) = \mathcal{D}^s_{q^s_{\ell_s}}(\mu(\rho^{i_j}))$.*

It can be easily seen that, for $q^s_{\ell_s} \in Q^s$ with $s = 1, \ldots, k$, the $(q^1_{\ell_1}, \cdots, q^k_{\ell_k})$-well-formedness relation is *transitive*.

Now it is possible to show that a track whose length exceeds a suitable exponential threshold, induces a shorter well-formed track. Such a contraction pattern represents a "basic step" in a contraction process which will allow us to prove the exponential small-model property for $\mathsf{A\overline{A}B\overline{B}}$. Let us consider an $\mathsf{A\overline{A}B\overline{B}}$ formula φ and let $r_1, \ldots, r_k$ be the RE's over $\mathcal{AP}$ in φ. Let $\mathcal{D}^1, \ldots, \mathcal{D}^k$ be the DFAs such that $\mathcal{L}(\mathcal{D}^t) = \mathcal{L}(r_t)$, for $t = 1, \ldots, k$, where $|Q^t| \leq 2^{2|r_t|}$ (see Remark 1). We denote $Q^1 \times \ldots \times Q^k$ by $Q(\varphi)$, and $\mathcal{D}^1, \ldots, \mathcal{D}^k$ by $\mathcal{D}(\varphi)$.

Proposition 4. *Let $\mathcal{K} = (\mathcal{AP}, S, R, \mu, s_0)$ be a finite Kripke structure, φ be an $\mathsf{A\overline{A}B\overline{B}}$ formula with RE's $r_1, \ldots, r_k$ over $\mathcal{AP}$, $\rho \in \mathrm{Trk}_{\mathcal{K}}$ be a track, and $(q^1, \cdots, q^k) \in Q(\varphi)$. There exists a track $\pi \in \mathrm{Trk}_{\mathcal{K}}$, which is $(q^1, \cdots, q^k)$-well-formed w.r.t. ρ, such that $|\pi| \leq |S| \cdot 2^{2\sum_{\ell=1}^{k} |r_\ell|}$.*

Proof. Let $\rho \in \mathrm{Trk}_{\mathcal{K}}$ with $|\rho| = n$. If $n \leq |S| \cdot 2^{2\sum_{\ell=1}^{k} |r_\ell|}$, the thesis trivially holds. Thus, let us assume $n > |S| \cdot 2^{2\sum_{\ell=1}^{k} |r_\ell|}$. We show that there exists a track which is $(q^1, \cdots, q^k)$-well-formed w.r.t. ρ, whose length is smaller than n. The number of possible (joint) configurations of the DFAs $\mathcal{D}(\varphi)$ is (at most) $|Q(\varphi)| \leq 2^{2|r_1|} \ldots 2^{2|r_k|} = 2^{2\sum_{\ell=1}^{k} |r_\ell|}$. Since $n > |S| \cdot 2^{2\sum_{\ell=1}^{k} |r_\ell|}$, there exists some state $s \in S$ occurring in ρ at least twice in the ρ-positions say $1 \leq l_1 < l_2 \leq |\rho|$, such that $\mathcal{D}^t_{q^t}(\mu(\rho^{l_1})) = \mathcal{D}^t_{q^t}(\mu(\rho^{l_2}))$, for all $t = 1, \ldots, k$. Let us consider $\pi = \rho(1, l_1) \star \rho(l_2, n)$. It is easy to see that $\pi \in \mathrm{Trk}_{\mathcal{K}}$, as $\rho(l_1) = \rho(l_2)$, and $|\pi| < n$. Moreover, π is $(q^1, \cdots, q^k)$-well-formed w.r.t. ρ (the corresponding positions are $i_j = j$ if $j \leq l_1$, and $i_j = j + (l_2 - l_1)$ otherwise). Now, if $|\pi| \leq |S| \cdot 2^{2\sum_{\ell=1}^{k} |r_\ell|}$, the thesis holds. Otherwise, the same basic step can be iterated a finite number of times: the thesis follows by transitivity of $(q^1, \cdots, q^k)$-well-formedness. $\square$

The next step is to determine some conditions for contracting tracks while preserving the equivalence w.r.t. the satisfiability of a considered $\mathsf{A\overline{A}B\overline{B}}$ formula. In the following we restrict ourselves to formulas in *negation normal form* (abbreviated NNF, and also known as *positive normal form*), namely, formulas where negation is applied only to atomic formulas (i.e., regular expressions)[3]. Any formula in $\mathsf{A\overline{A}B\overline{B}}$ can be converted (in linear time) into an equivalent one in NNF, having at most double length (by using De Morgan's laws and duality of HS modalities).

[3]Not to be confused with the *negation form* of the previous section.

For a track ρ and a formula φ of $\mathsf{A\overline{A}B\overline{B}}$ (in NNF), we fix some special ρ-positions, called *witness positions*, each one corresponding to the minimal prefix of ρ which satisfies a formula ψ occurring in φ as a subformula of the form $\langle \mathrm{B} \rangle\, \psi$ (provided that $\langle \mathrm{B} \rangle\, \psi$ is satisfied by ρ). As we shall prove in Theorem 1, when a contraction is performed (as in the proof of Proposition 4) in between a pair of *consecutive* witness positions (hence no witness position is removed by any contraction), we get a track induced by ρ which is equivalent w.r.t. the satisfiability of φ.

Definition 5 (Witness positions)**.** *Let ρ be a track of $\mathcal{K}$ and φ be a formula of $\mathsf{A\overline{A}B\overline{B}}$. Let us denote by $B(\varphi,\rho)$ the set of subformulas $\langle \mathrm{B} \rangle\, \psi$ of φ such that $\mathcal{K},\rho \models \langle \mathrm{B} \rangle\, \psi$. The* set $Wt(\varphi,\rho)$ of witness positions of ρ for φ *is the* minimal *set of ρ-positions satisfying the following constraint: for each $\langle \mathrm{B} \rangle\, \psi \in B(\varphi,\rho)$, the smallest ρ-position $i < |\rho|$ such that $\mathcal{K},\rho^i \models \psi$ belongs to $Wt(\varphi,\rho)$*[4].

The cardinality of $B(\varphi,\rho)$ and of $Wt(\varphi,\rho)$ is at most $|\varphi|$–1.

Theorem 1 (Exponential small-model for $\mathsf{A\overline{A}B\overline{B}}$)**.** *Let $\mathcal{K} = (\mathcal{AP}, S, R, \mu, s_0)$, $\sigma,\rho \in \mathrm{Trk}_{\mathcal{K}}$, and φ be an $\mathsf{A\overline{A}B\overline{B}}$ formula in NNF, with* RE*'s $r_1,\ldots,r_u$ over $\mathcal{AP}$, such that $\mathcal{K},\sigma\star\rho \models \varphi$. Then, there is $\pi \in \mathrm{Trk}_{\mathcal{K}}$, induced by ρ, such that $\mathcal{K},\sigma\star\pi \models \varphi$ and $|\pi| \le |S|\cdot(|\varphi|+1)\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|}$.*

Notice that Theorem 1 holds in particular if $|\sigma| = 1$, and thus $\sigma\star\rho = \rho$ and $\sigma\star\pi = \pi$. In this case, if $\mathcal{K},\rho \models \varphi$, then $\mathcal{K},\pi \models \varphi$, where π is induced by ρ and $|\pi| \le |S|\cdot(|\varphi|+1)\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|}$. The more general assertion of Theorem 1 is motivated by technical reasons.

Proof. Let $Wt(\varphi,\sigma\star\rho)$ be the set of witness positions of $\sigma\star\rho$ for φ. Let $\{i_1,\ldots,i_k\}$ be the ordering of $Wt(\varphi,\sigma\star\rho)$ such that $i_1 < \ldots < i_k$. Let $i_0 = 1$ and $i_{k+1} = |\sigma\star\rho|$. Hence, $1 = i_0 \le i_1 < \ldots < i_k < i_{k+1} = |\sigma\star\rho|$.

If the length of ρ is at most $|S|\cdot(|\varphi|+1)\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|}$, the thesis trivially holds. Let us assume that $|\rho| > |S|\cdot(|\varphi|+1)\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|}$. We show that there exists a track π induced by ρ, with $|\pi| < |\rho|$, such that $\mathcal{K},\sigma\star\pi \models \varphi$.

W.l.o.g., we can assume that $i_0 \le i_1 < \ldots < i_{j-1}$, for some $j \ge 1$, are σ-positions (while $i_j < \ldots < i_{k+1}$ are $(\sigma\star\rho)$-positions not in σ). We claim that either (i) there exists $t \in [j,k]$ such that $i_{t+1} - i_t > |S|\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|}$ or (ii) $|(\sigma\star\rho)(|\sigma|,i_j)| > |S|\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|}$. By way of contradiction, suppose that neither (i) nor (ii) holds. We need to distinguish two cases. If $\sigma\star\rho = \rho$, then $|\rho| = (i_{k+1} - i_0) + 1 \le (k+1)\cdot|S|\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|} + 1$; otherwise $(|\rho| < |\sigma\star\rho|)$, $|\rho| = (i_{k+1} - i_j) + |(\sigma\star\rho)(|\sigma|,i_j)| \le k\cdot|S|\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|} + |S|\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|} \le (k+1)\cdot|S|\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|}$. The contradiction follows since $(k+1)\cdot|S|\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|} + 1 \le |\varphi|\cdot|S|\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|} + 1 \le |S|\cdot(|\varphi|+1)\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|}$.

Let us define $(\alpha,\beta) = (i_t, i_{t+1})$ in case (i), and $(\alpha,\beta) = (|\sigma|, i_j)$ in case (ii). Moreover let $\rho' = \rho(\alpha,\beta)$. In both the cases, we have $|\rho'| > |S|\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|}$. By Proposition 4, there exists a track π' of $\mathcal{K}$, $(q^1,\cdots,q^u)$-well-formed with respect to ρ', such that $|\pi'| \le |S|\cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|} < |\rho'|$, where we choose $q^x = \mathcal{D}^x(\mu((\sigma\star\rho)^{\alpha-1}))$ for $x = 1,\ldots,u$ (as a particular case we set q_x as the initial state of $\mathcal{D}^x$ if $\alpha = 1$). Let π be the track induced by ρ obtained by replacing the subtrack ρ' of ρ with π'. Since $|\pi| < |\rho|$, it remains to prove that $\mathcal{K},\sigma\star\pi \models \varphi$.

Let us denote $\sigma\star\pi$ by $\overline{\pi}$ and $\sigma\star\rho$ by $\overline{\rho}$. Moreover, let $H : [1,|\overline{\pi}|] \to [1,|\overline{\rho}|]$ be the function mapping positions of $\overline{\pi}$ into positions of $\overline{\rho}$ in this way: positions "outside" π' (i.e., outside the interval $[\alpha,\alpha+|\pi'|-1]$) are mapped into their original position in $\overline{\rho}$; positions "inside" π' (i.e., in $[\alpha,\alpha+|\pi'|-1]$) are mapped to the corresponding position in ρ' (exploiting well-formedness of π' w.r. to ρ'). Formally, H is defined as:

$$H(m) = \begin{cases} m & \text{if} \quad m < \alpha \\ \alpha + \ell_{m-\alpha+1} - 1 & \text{if} \quad \alpha \le m < \alpha + |\pi'| \\ m + (|\rho'| - |\pi'|) & \text{if} \quad m \ge \alpha + |\pi'| \end{cases} \qquad (1)$$

where ℓ_m is the ρ'-position corresponding to the π'-position m. It is easy to check that H satisfies the following properties:

1) H is strictly monotonic, i.e., for all $j,j' \in [1,|\overline{\pi}|]$, $j < j'$ iff $H(j) < H(j')$;
2) for all $j \in [1,|\overline{\pi}|]$, $\overline{\pi}(j) = \overline{\rho}(H(j))$;
3) $H(1) = 1$ and $H(|\overline{\pi}|) = |\overline{\rho}|$;
4) $Wt(\varphi,\overline{\rho}) \subseteq \{H(j) \mid j \in [1,|\overline{\pi}|]\}$;
5) for each $j \in [1,|\overline{\pi}|]$ and $x = 1,\ldots,u$, $\mathcal{D}^x(\mu(\overline{\pi}^j)) = \mathcal{D}^x(\mu(\overline{\rho}^{H(j)}))$.

We only comment on Property 5. The property holds for $j \in [1,\alpha-1]$, as $\overline{\pi}^j = \overline{\rho}^{H(j)} = \overline{\rho}^j$. For $j \in [\alpha,\alpha+|\pi'|-1]$, $\mathcal{D}^x(\mu(\overline{\pi}^j)) = \mathcal{D}^x(\mu(\overline{\rho}^{H(j)}))$ follows from the well-formedness hypothesis. Finally, being $\overline{\rho}(\beta,|\overline{\rho}|) = \overline{\pi}(\alpha+|\pi'|-1,|\overline{\pi}|)$ and $\mathcal{D}^x(\mu(\overline{\pi}^{\alpha+|\pi'|-1})) = \mathcal{D}^x(\mu(\overline{\rho}^{\beta}))$, the property holds also for $j \in [\alpha+|\pi'|,|\overline{\pi}|]$.

The statement $\mathcal{K},\overline{\pi} \models \varphi$ is an immediate consequence of the following claim, considering that $H(|\overline{\pi}|) = |\overline{\rho}|$, $\mathcal{K},\overline{\rho} \models \varphi$, $\overline{\rho}^{|\overline{\rho}|} = \overline{\rho}$, and $\overline{\pi}^{|\overline{\pi}|} = \overline{\pi}$.

Claim 1. *For all $j \in [1,|\overline{\pi}|]$, all subformulas ψ of φ, and all $\xi \in \mathrm{Trk}_{\mathcal{K}}$, if $\mathcal{K},\overline{\rho}^{H(j)}\star\xi \models \psi$ then $\mathcal{K},\overline{\pi}^j\star\xi \models \psi$.*

Proof. Assume that $\mathcal{K},\overline{\rho}^{H(j)}\star\xi \models \psi$. Note that $\overline{\rho}^{H(j)}\star\xi$ is defined iff $\overline{\pi}^j\star\xi$ is defined. We prove by induction on the structure of ψ that $\mathcal{K},\overline{\pi}^j\star\xi \models \psi$. Since φ is in NNF, only the following cases can occur:

- $\psi = r_t$ or $\psi = \neg r_t$ where r_t is some RE over $\mathcal{AP}$. By Property 5 of H, $\mathcal{D}^t(\mu(\overline{\pi}^j)) = \mathcal{D}^t(\mu(\overline{\rho}^{H(j)}))$, thus $\mathcal{D}^t(\mu(\overline{\pi}^j\star\xi)) = \mathcal{D}^t(\mu(\overline{\rho}^{H(j)}\star\xi))$. It follows that $\mathcal{K},\overline{\pi}^j\star\xi \models r_t$ iff $\mathcal{K},\overline{\rho}^{H(j)}\star\xi \models r_t$, and the result holds.
- $\psi = \theta_1 \wedge \theta_2$ or $\psi = \theta_1 \vee \theta_2$, for some $\mathsf{A\overline{A}B\overline{B}}$ formulas θ_1 and θ_2. The result holds by the inductive hypothesis.
- $\psi = [B]\theta$. We need to show that for each proper prefix η of $\overline{\pi}^j\star\xi$, $\mathcal{K},\eta \models \theta$. We distinguish two cases:
  - η is *not* a proper prefix of $\overline{\pi}^j$. Hence, η is of the form $\overline{\pi}^j\star\xi^h$ for some $h \in [1,|\xi|-1]$. Since $\mathcal{K},\overline{\rho}^{H(j)}\star\xi \models [B]\theta$, then $\mathcal{K},\overline{\rho}^{H(j)}\star\xi^h \models \theta$. By the inductive hypothesis, $\mathcal{K},\overline{\pi}^j\star\xi^h \models \theta$.
  - η is a proper prefix of $\overline{\pi}^j$. Hence, $\eta = \overline{\pi}^h$ for some $h \in [1,j-1]$. By Property 1 of H, $H(h) < H(j)$, and

[4]Note that such a ρ-position exists by definition of $B(\varphi,\rho)$.

since $\mathcal{K}, \overline{\rho}^{H(j)} \star \xi \models [B]\theta$, we have that $\mathcal{K}, \overline{\rho}^{H(h)} \models \theta$. By the inductive hypothesis, $\mathcal{K}, \overline{\pi}^h \models \theta$.

Therefore, $\mathcal{K}, \overline{\pi}^j \star \xi \models [B]\theta$.

- $\psi = \langle \mathrm{B} \rangle \theta$. We need to show that there exists a proper prefix of $\overline{\pi}^j \star \xi$ satisfying θ. Since $\mathcal{K}, \overline{\rho}^{H(j)} \star \xi \models \psi$, there exists a proper prefix η' of $\overline{\rho}^{H(j)} \star \xi$ such that $\mathcal{K}, \eta' \models \theta$. We distinguish two cases:
  - η' is *not* a proper prefix of $\overline{\rho}^{H(j)}$. Hence, η' is of the form $\overline{\rho}^{H(j)} \star \xi^h$ for some $h \in [1, |\xi|-1]$. By the inductive hypothesis, $\mathcal{K}, \overline{\pi}^j \star \xi^h \models \theta$, and $\mathcal{K}, \overline{\pi}^j \star \xi \models \langle \mathrm{B} \rangle \theta$.
  - η' is a proper prefix of $\overline{\rho}^{H(j)}$. Hence, $\eta' = \overline{\rho}^i$ for some $i \in [1, H(j)-1]$, and $\mathcal{K}, \overline{\rho}^i \models \theta$. Let i' be the smallest position of $\overline{\rho}$ such that $\mathcal{K}, \overline{\rho}^{i'} \models \theta$. Hence $i' \le i$ and, by Definition 5, $i' \in Wt(\varphi, \overline{\rho})$. By Property 4 of H, $i' = H(h)$ for some $\overline{\pi}$-position h. Since $H(h) < H(j)$, it holds that $h < j$ (Property 1). By the inductive hypothesis, $\mathcal{K}, \overline{\pi}^h \models \theta$, and $\mathcal{K}, \overline{\pi}^j \star \xi \models \langle \mathrm{B} \rangle \theta$.
- $\psi = [\overline{B}]\theta$ or $\psi = \langle \overline{\mathrm{B}} \rangle \theta$. The result holds as a direct consequence of the inductive hypothesis.
- $\psi = [A]\theta$, $\psi = \langle \mathrm{A} \rangle \theta$, $\psi = [\overline{A}]\theta$ or $\psi = \langle \overline{\mathrm{A}} \rangle \theta$. Since $\overline{\pi}^j \star \xi$ and $\overline{\rho}^{H(j)} \star \xi$ start at the same state and lead to the same state (by Properties 2 and 3 of H), the result trivially follows, concluding the proof of the claim. □

We have proved that $\mathcal{K}, \overline{\pi} \models \varphi$, with $|\pi| < |\rho|$. If $|\pi| \le |S| \cdot (|\varphi|+1) \cdot 2^{2\sum_{\ell=1}^{u} |r_\ell|}$, the thesis hold. Otherwise, we can iterate the above contraction (a finite number of times) until the bound is achieved. □

The proved exponential small-model allows us to devise a trivial *exponential working space* algorithm for $\mathsf{A\overline{A}B\overline{B}}$ (and $\mathsf{A\overline{A}E\overline{E}}$) (actually we shall present a polynomial space algorithm in the next sections), which basically unravels the Kripke structure and checks all the subformulas of the input formula. At every step it can consider tracks not longer than $O(|S| \cdot |\varphi| \cdot 2^{2\sum_{\ell=1}^{u} |r_\ell|})$.

The following example shows that the exponential small-model is strict, that is, there exists a formula and a Kripke structure, such that the shortest track satisfying the formula has exponential length in the size of the formula. This is the case even for pure propositional formulas.

Example 1. *Let pr_i be the i-th smallest prime. It is well-known that $pr_i \in O(i \log i)$. Let $w^{\otimes k}$ denote the string obtained by concatenating k times w. Let us fix some $n \in \mathbb{N}$, and let $\mathcal{K} = (\{p\}, \{s\}, R, \mu, s)$ be the trivial Kripke structure having only one state with a self-loop, where $R = \{(s,s)\}$, and $\mu(s) = \{p\}$. The shortest track satisfying $\psi = \bigwedge_{i=1}^{n} (p^{\otimes (pr_i)})^*$ is $\rho = s^{\otimes (pr_1 \cdots pr_n)}$, since its length is the least common multiple of $pr_1, \ldots, pr_n$, which is indeed $pr_1 \cdots pr_n$. It is immediate to check that the length of ψ is $O(n \cdot pr_n) = O(n^2 \log n)$. On the other hand, the length of ρ is $pr_1 \cdots pr_n \ge 2^n$.*

In the remaining part of the paper we exploit the exponential small-model for $\mathsf{A\overline{A}B\overline{B}}$ and $\mathsf{A\overline{A}E\overline{E}}$ to prove the complexity (**PSPACE**-completeness) of the MC problem for the two symmetrical fragments. In particular, in the following section we provide a **PSPACE** MC algorithm for $\mathsf{B\overline{B}}$ (resp., $\mathsf{E\overline{E}}$). Then we show that the *meets* and *met-by* modalities A and $\mathsf{\overline{A}}$ can be suitably encoded by using regular expressions, thus proving they do not increase the complexity of $\mathsf{B\overline{B}}$ (resp., $\mathsf{E\overline{E}}$).

VI. A PSPACE MC ALGORITHM FOR $\mathsf{B\overline{B}}$

In this section we shall describe a **PSPACE** MC algorithm for $\mathsf{B\overline{B}}$ formulas. In the proposed algorithm we assume w.l.o.g. that the processed formulas do not contain occurrences of the universal modalities $[B]$ and $[\overline{B}]$. Moreover, for a formula ψ, we denote by $\mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi) = \{\varphi \mid \langle \mathrm{B} \rangle \varphi$ is a subformula of $\psi\}$; Φ represents the overall formula to be checked, while the parametric formula ψ ranges over its subformulas.

Due to the result of the previous section, the algorithm can consider only tracks having length bounded by the exponential small-model property. Notice that an algorithm required to work in polynomial space cannot explicitly store the DFAs for the regular expressions occurring in Φ (their states are *exponentially* many in the length of the associated regular expressions). For this reason, while checking a formula against a track, the algorithm just stores the *current states* of the computations of the DFAs associated with the regular expressions in Φ, from the respective initial states (in the following such states are denoted—with a little abuse of notation—again by $\mathcal{D}(\Phi)$, and called the *"current configuration" of the DFAs*) and calculates on-the-fly the successor states in the DFAs, once they have read some state of $\mathcal{K}$ used to extend the considered track (this can be done by exploiting a *succinct* encoding of the NFAs for the regular expressions of Φ, see Remark 1).

A call to the recursive procedure $\mathtt{Check}(\mathcal{K}, \psi, s, G, \mathcal{D}(\Phi))$ (Algorithm 1) checks the satisfiability of a subformula ψ of Φ with respect to any track ρ fulfilling the following conditions: (1) $G \subseteq \mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi)$ is the set of formulas that hold true on at least a prefix of ρ; (2) after reading $\mu(\rho(1, |\rho|-1))$ the current configuration of the DFAs for the regular expressions of Φ is $\mathcal{D}(\Phi)$; (3) the last state of ρ is s. Intuitively, since the algorithm cannot store the already checked portion of a track (whose length could be exponential), the relevant information is *summarized* in a triple $(G, \mathcal{D}(\Phi), s)$. In the following the set of all possible summarizing triples $(\overline{G}, \overline{\mathcal{D}(\Phi)}, \overline{s})$, where $\overline{G} \subseteq \mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi)$, $\overline{\mathcal{D}(\Phi)}$ is any current configuration of the DFAs for the regular expressions of Φ, and $\overline{s}$ is a state of $\mathcal{K}$, is denoted by $\mathsf{Conf}(\mathcal{K}, \psi)$.

Let us consider in detail the body of the procedure. First of all, $\mathtt{advance}(\mathcal{D}(\Phi), \mu(s))$, invoked at line 2, updates the current configuration of the DFAs after reading the symbol $\mu(s)$. If ψ is an interval property associated with a regular expression r (lines 1–5), we just check whether the (computation of the) DFA associated with r is in a final state (i.e., the summarized track is accepted). Boolean connectives are easily dealt with recursively (lines 6–9). If ψ has the form $\langle \mathrm{B} \rangle \psi'$ (lines 10–14), then ψ' has to hold over a proper prefix of the summarized track, namely, ψ' must belong to G.

The only involved case is the one where $\psi = \langle \overline{\mathrm{B}} \rangle \psi'$ (lines 15–19). To deal with this, we have to unravel the

Algorithm 1 `Check`$(\mathcal{K}, \psi, s, G, \mathcal{D}(\Phi))$

1: **if** $\psi = r$ **then** ◁ *r is a regular expression*
2: **if** the current state of the DFA for r in `advance`$(\mathcal{D}(\Phi), \mu(s))$ is final **then**
3: **return** $\top$
4: **else**
5: **return** $\bot$
6: **else if** $\psi = \neg\psi'$ **then**
7: **return not** `Check`$(\mathcal{K}, \psi', s, G, \mathcal{D}(\Phi))$
8: **else if** $\psi = \psi_1 \wedge \psi_2$ **then**
9: **return** `Check`$(\mathcal{K}, \psi_1, s, G \cap \mathsf{Subf}_{\langle B \rangle}(\psi_1), \mathcal{D}(\Phi))$ **and** `Check`$(\mathcal{K}, \psi_2, s, G \cap \mathsf{Subf}_{\langle B \rangle}(\psi_2), \mathcal{D}(\Phi))$
10: **else if** $\psi = \langle B \rangle\, \psi'$ **then**
11: **if** $\psi' \in G$ **then**
12: **return** $\top$
13: **else**
14: **return** $\bot$
15: **else if** $\psi = \langle \overline{B} \rangle\, \psi'$ **then**
16: **for** each $b \in \{1, \ldots, |S| \cdot (2|\psi'| + 1) \cdot 2^{2 \sum_{\ell=1}^{u} |r_\ell|} - 1\}$ and each $(G', \mathcal{D}(\Phi)', s') \in \mathsf{Conf}(\mathcal{K}, \psi)$ **do** ◁ *$r_1, .., r_u$ are the r.e. of ψ'*
17: **if** `Reach`$(\mathcal{K}, \psi', (G, \mathcal{D}(\Phi), s), (G', \mathcal{D}(\Phi)', s'), b)$ **and** `Check`$(\mathcal{K}, \psi', s', G', \mathcal{D}(\Phi)')$ **then**
18: **return** $\top$
19: **return** $\bot$

Algorithm 2 `Reach`$(\mathcal{K}, \psi, (G_1, \mathcal{D}(\Phi)_1, s_1), (G_2, \mathcal{D}(\Phi)_2, s_2), b)$

1: **if** $b = 1$ **then**
2: **return** `Compatible`$(\mathcal{K}, \psi, (G_1, \mathcal{D}(\Phi)_1, s_1), (G_2, \mathcal{D}(\Phi)_2, s_2))$
3: **else** ◁ $b \geq 2$
4: $b' \leftarrow \lfloor b/2 \rfloor$
5: **for** each $(G_3, \mathcal{D}(\Phi)_3, s_3) \in \mathsf{Conf}(\mathcal{K}, \psi)$ **do**
6: **if** `Reach`$(\mathcal{K}, \psi, (G_1, \mathcal{D}(\Phi)_1, s_1), (G_3, \mathcal{D}(\Phi)_3, s_3), b')$ **and** `Reach`$(\mathcal{K}, \psi, (G_3, \mathcal{D}(\Phi)_3, s_3), (G_2, \mathcal{D}(\Phi)_2, s_2), b - b')$ **then**
7: **return** $\top$
8: **return** $\bot$

Algorithm 3 `Compatible`$(\mathcal{K},\psi,(G_1,\mathcal{D}(\Phi)_1,s_1),(G_2,\mathcal{D}(\Phi)_2,s_2))$

1: **if** $(s_1, s_2) \in R$ **and** `advance`$(\mathcal{D}(\Phi)_1, \mu(s_1)) = \mathcal{D}(\Phi)_2$ **and** $G_1 \subseteq G_2$ **then**
2: **for** each $\varphi \in (G_2 \setminus G_1)$ **do**
3: $G \leftarrow G_1 \cap \mathsf{Subf}_{\langle B \rangle}(\varphi)$
4: **if** `Check`$(\mathcal{K}, \varphi, s_1, G, \mathcal{D}(\Phi)_1) = \bot$ **then**
5: **return** $\bot$
6: **for** each $\varphi \in (\mathsf{Subf}_{\langle B \rangle}(\psi) \setminus G_2)$ **do**
7: $G \leftarrow G_1 \cap \mathsf{Subf}_{\langle B \rangle}(\varphi)$
8: **if** `Check`$(\mathcal{K}, \varphi, s_1, G, \mathcal{D}(\Phi)_1) = \top$ **then**
9: **return** $\bot$
10: **return** $\top$
11: **else**
12: **return** $\bot$

Kripke structure $\mathcal{K}$ to find an *extension* ρ' of ρ, summarized by the triple $(G', \mathcal{D}(\Phi)', s')$, satisfying ψ'. The idea is checking whether or not there exists a summarized track $(G', \mathcal{D}(\Phi)', s')$, suitably extending $(G, \mathcal{D}(\Phi), s)$, namely, such that: (1) $\mathcal{D}(\Phi)'$ and s' are *synchronously* reachable from $\mathcal{D}(\Phi)$ and s, resp.; (2) $G' \supseteq G$ contains all the formulas of $\mathsf{Subf}_{\langle B \rangle}(\psi')$ satisfied by some prefixes of the extension; (3) the extension $(G', \mathcal{D}(\Phi)', s')$ satisfies ψ'. In order to check point (1), i.e., synchronous reachability, we can exploit the exponential small-model property and consider only the unravelling of $\mathcal{K}$ starting from s having depth at most $|S| \cdot (2|\psi'| + 1) \cdot 2^{2 \sum_{\ell=1}^{u} |r_\ell|} - 1$.[5] The check of (1) and (2) is performed by the procedure `Reach` (Algorithm 2), which accepts as input two summarized tracks and a bound b on the depth of the unravelling of $\mathcal{K}$. The proposed reachability algorithm is reminiscent of the binary reachability technique in the proof of Savitch's theorem [GJ79].

`Reach` proceeds recursively (lines 3–8) by halving at each step the value b of the length bound, until it gets called over two states s_1 and s_2 which are adjacent in a track. At each halving step, an intermediate summarizing triple is generated to be associated with the split point. At the base of recursion (for $b = 1$, lines 1–2), the auxiliary procedure `Compatible` (Algorithm 3) is invoked. At line 1, `Compatible` checks whether there is an edge between s_1 and s_2 ($(s_1, s_2) \in R$), and if, at the considered step, the current configuration of the DFAs $\mathcal{D}(\Phi)_1$ is transformed into the configuration $\mathcal{D}(\Phi)_2$ (i.e., s_2 and $\mathcal{D}(\Phi)_2$ are synchronously reachable from s_1 and $\mathcal{D}(\Phi)_1$). At lines 2–9, `Compatible` checks that each formula φ in $(G_2 \setminus G_1)$, where $G_2 \supseteq G_1$, is satisfied by a track summarized by $(G_1, \mathcal{D}(\Phi)_1, s_1)$ (lines 2–5). Intuitively, $(G_1, \mathcal{D}(\Phi)_1, s_1)$ summarizes the maximal prefix of $(G_2, \mathcal{D}(\Phi)_2, s_2)$, and thus a subformula satisfied by a prefix of a track summarized by $(G_2, \mathcal{D}(\Phi)_2, s_2)$ either belongs to G_1 or it is satisfied by the track summarized by $(G_1, \mathcal{D}(\Phi)_1, s_1)$. Moreover, (lines 6–9) `Compatible` checks that G_2 is maximal (i.e., no subformula that must be in G_2 has been forgot).

Note that by exploiting this binary reachability technique, the recursion depth of `Reach` is logarithmic in the length of the track to be visited, hence it can use only polynomial space.

The next theorem establishes the soundness of `Check`.

Theorem 2. *Let Φ be a $\mathsf{B\overline{B}}$ formula, ψ be a subformula of Φ, and $\rho \in \mathrm{Trk}_{\mathcal{K}}$ be a track with $s = \mathrm{lst}(\rho)$. Let G be the subset of formulas in $\mathsf{Subf}_{\langle B \rangle}(\psi)$ that hold on some proper prefix of ρ. Let $\mathcal{D}(\Phi)$ be the current configuration of the DFAs associated with the regular expressions in Φ after reading $\mu(\rho(1, |\rho|-1))$. Then `Check`$(\mathcal{K}, \psi, s, G, \mathcal{D}(\Phi)) = \top \iff \mathcal{K}, \rho \models \psi$.*

The proof is reported in Appendix A. Finally, the MC procedure for $\mathsf{B\overline{B}}$ formulas is reported in Algorithm 4. The main procedure `CheckAux`$(\mathcal{K}, \Phi)$ starts by constructing the NFAs and the initial states of the DFAs for the regular expressions of Φ[6]. Then `CheckAux` invokes the procedure `Check` two times: the former to check the special case of the track s_0 (consisting of the initial state of $\mathcal{K}$ only), and the latter for considering any right-extensions of s_0 (i.e., all the initial tracks having length at least 2).

Theorem 3. *Let $\mathcal{K} = (\mathcal{AP}, S, R, \mu, s_0)$ be a finite Kripke structure, and Φ be a $\mathsf{B\overline{B}}$ formula. Then `CheckAux`$(\mathcal{K}, \Phi)$*

[5] The factor 2 in front of $|\psi'|$ is due to the fact that the exponential small-model for $\mathsf{A\overline{A}B\overline{B}}$ requires a formula in NNF.

[6] Note that the NFAs and DFAs for the regular expressions of $\langle \overline{B} \rangle \neg\Phi$ and $\neg\Phi$ are exactly the same as those for Φ: thus $\mathcal{D}(\Phi)_0 = \mathcal{D}(\langle \overline{B} \rangle \neg\Phi)_0 = \mathcal{D}(\neg\Phi)_0$, hence Th. 2 applies at line 2, for both the invocations of `Check`.

Algorithm 4 `CheckAux`$(\mathcal{K}, \Phi)$

1: `create`$(\mathcal{D}(\Phi)_0)$ $\triangleleft$ *Creates the (succinct) NFAs and the initial states of the DFAs for all the regular expressions in* Φ
2: **if** `Check`$(\mathcal{K}, \neg\Phi, s_0, \emptyset, \mathcal{D}(\Phi)_0)$ **or** `Check`$(\mathcal{K}, \langle \overline{\mathrm{B}} \rangle \neg\Phi, s_0, \emptyset, \mathcal{D}(\Phi)_0)$ **then**
3: **return** $\perp$
4: **else**
5: **return** $\top$

returns $\top$ *if and only if* $\mathcal{K} \models \Phi$.

Proof. If $\mathcal{K} \models \Phi$, then for all $\rho \in \mathrm{Trk}_{\mathcal{K}}$ with $\mathrm{fst}(\rho) = s_0$, we have $\mathcal{K}, \rho \models \Phi$, hence $\mathcal{K}, s_0 \models \Phi$, and $\mathcal{K}, s_0 \cdot \rho' \models \Phi$ for all $s_0 \cdot \rho' \in \mathrm{Trk}_{\mathcal{K}}$, thus $\mathcal{K}, s_0 \models [\overline{B}]\Phi$, namely, $\mathcal{K}, s_0 \not\models \langle \overline{\mathrm{B}} \rangle \neg\Phi$. By Theorem 2, `Check`$(\mathcal{K}, \neg\Phi, s_0, \emptyset, \mathcal{D}(\Phi)_0) = \perp$ and `Check`$(\mathcal{K}, \langle \overline{\mathrm{B}} \rangle \neg\Phi, s_0, \emptyset, \mathcal{D}(\Phi)_0) = \perp$ implying that `CheckAux`$(\mathcal{K}, \Phi)$ returns $\top$.

Conversely, if the procedure `CheckAux`$(\mathcal{K}, \Phi)$ returns $\top$, then it must be `Check`$(\mathcal{K}, \neg\Phi, s_0, \emptyset, \mathcal{D}(\Phi)_0) = \perp$ and `Check`$(\mathcal{K}, \langle \overline{\mathrm{B}} \rangle \neg\Phi, s_0, \emptyset, \mathcal{D}(\Phi)_0) = \perp$. By Theorem 2 applied to the track $\rho = s_0$, we have $\mathcal{K}, s_0 \not\models \neg\Phi$ and $\mathcal{K}, s_0 \not\models \langle \overline{\mathrm{B}} \rangle \neg\Phi$, and, therefore, $\mathcal{K} \models \Phi$. $\square$

Corollary 2. *The MC problem for* $\mathsf{B\overline{B}}$ *formulas over finite Kripke structures is in* $\mathbf{PSPACE}$.

Proof. The procedure `CheckAux` decides the problem using *polynomial work space* due to the following facts:

- the number of simultaneously active recursive calls of `Check` is $O(|\Phi|)$ (depending on the depth of Φ);
- for any call of `Check` the used space (in bits) is

$$O\Big((|\Phi| + |S| + \sum_{\ell=1}^{u} |r_\ell| + \underbrace{\log(|S| \cdot |\Phi| \cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|})}_{(1)} + \underbrace{(|\Phi| + |S| + \sum_{\ell=1}^{u} |r_\ell|)}_{(2)} \cdot \underbrace{\log(|S| \cdot |\Phi| \cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|})}_{(3)}\Big)$$

  where $r_1, \dots, r_u$ are the regular expressions of Φ, and S the states of $\mathcal{K}$. In particular, (1) $O(\log(|S| \cdot |\Phi| \cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|}))$ bits are used for the bound b on the track length, (3) for *each subformula* $\langle \overline{\mathrm{B}} \rangle \psi'$ of Φ at most $O(\log(|S| \cdot |\Phi| \cdot 2^{2\sum_{\ell=1}^{u}|r_\ell|}))$ recursive calls of `Reach` may be simultaneously active (the recursion depth of `Reach` is logarithmic in the value of b), and (2) each call of `Reach` requires $O(|\Phi| + |S| + \sum_{\ell=1}^{u} |r_\ell|)$ bits. $\square$

Finally, since a Kripke structure can be unravelled against the direction of its edges and, given a regular language $\mathcal{L}$, its reversed version $\mathcal{L}^{\mathrm{Rev}} = \{w(|w|) \cdots w(1) \mid w \in \mathcal{L}\}$ is regular as well, the proposed algorithm can be easily modified to deal with formulas of the symmetrical fragment $\mathsf{E\overline{E}}$. Thus also the MC problem for $\mathsf{E\overline{E}}$ can be proved to be in $\mathbf{PSPACE}$.

VII. MC for $\mathsf{A\overline{A}B\overline{B}}$ is $\mathbf{PSPACE}$-complete

We now show that the algorithm `CheckAux`, devised in the previous section, can be used as a basic engine to design a $\mathbf{PSPACE}$ MC algorithm for $\mathsf{A\overline{A}B\overline{B}}$. The idea is that, being the proposition letters related with regular expressions, the modalities $\langle \mathrm{A} \rangle$ and $\langle \overline{\mathrm{A}} \rangle$ do not augment the expressiveness of the fragment $\mathsf{B\overline{B}}$. In particular, we shall show how $\langle \mathrm{A} \rangle$ and $\langle \overline{\mathrm{A}} \rangle$, occurring in an $\mathsf{A\overline{A}B\overline{B}}$ formula, can suitably be "absorbed" and replaced by fresh proposition letters.

By definition, $\mathcal{K}, \rho \models \langle \mathrm{A} \rangle \psi$ if and only if there exists a track $\tilde{\rho} \in \mathrm{Trk}_{\mathcal{K}}$ such that $\mathrm{lst}(\rho) = \mathrm{fst}(\tilde{\rho})$ and $\mathcal{K}, \tilde{\rho} \models \psi$. An immediate consequence is that, for any $\rho' \in \mathrm{Trk}_{\mathcal{K}}$ with $\mathrm{lst}(\rho) = \mathrm{lst}(\rho')$, $\mathcal{K}, \rho \models \langle \mathrm{A} \rangle \psi \iff \mathcal{K}, \rho' \models \langle \mathrm{A} \rangle \psi$. Analogous considerations can be done for the symmetrical modality $\langle \overline{\mathrm{A}} \rangle$ with respect to initial states of tracks. In general, if two tracks have the same final state (resp., first state), either both of them satisfy a formula $\langle \mathrm{A} \rangle \psi$ (resp., $\langle \overline{\mathrm{A}} \rangle \psi$), or none of them does. Therefore, for a formula $\langle \mathrm{A} \rangle \psi$ (resp., $\langle \overline{\mathrm{A}} \rangle \psi$), we can determine the subset $S_{\langle \mathrm{A} \rangle \psi}$ (resp., $S_{\langle \overline{\mathrm{A}} \rangle \psi}$) of the set of states S of the Kripke structure such that, for any $\rho \in \mathrm{Trk}_{\mathcal{K}}$, $\mathcal{K}, \rho \models \langle \mathrm{A} \rangle \psi$ (resp., $\mathcal{K}, \rho \models \langle \overline{\mathrm{A}} \rangle \psi$) if and only if $\mathrm{lst}(\rho) \in S_{\langle \mathrm{A} \rangle \psi}$ (resp., $\mathrm{fst}(\rho) \in S_{\langle \overline{\mathrm{A}} \rangle \psi}$).

Now, the idea is that we can identify each state $s \in S$ exploiting a set of fresh proposition letters $\{q_s \mid s \in S\}$; then we define, for a subformula $\langle \mathrm{A} \rangle \psi$ (resp., $\langle \overline{\mathrm{A}} \rangle \psi$), a regular expression $r_{\langle \mathrm{A} \rangle \psi}$ (resp., $r_{\langle \overline{\mathrm{A}} \rangle \psi}$) characterizing the set of tracks which model the subformula, and finally we replace any occurrence of $\langle \mathrm{A} \rangle \psi$ (resp., $\langle \overline{\mathrm{A}} \rangle \psi$) by a fresh interval property associated with this regular expression.

More formally, instead of $\mathcal{K} = (\mathcal{AP}, S, R, \mu, s_0)$, we consider $\mathcal{K}' = (\mathcal{AP}', S, R, \mu', s_0)$, with $\mathcal{AP}' := \mathcal{AP} \cup \{q_s \mid s \in S\}$ and $\mu'(s) = \{q_s\} \cup \mu(s)$ for any $s \in S$. For the formulas $\langle \mathrm{A} \rangle \psi$ and $\langle \overline{\mathrm{A}} \rangle \psi$, the regular expressions $r_{\langle \mathrm{A} \rangle \psi}$ and $r_{\langle \overline{\mathrm{A}} \rangle \psi}$ are:

$$r_{\langle \mathrm{A} \rangle \psi} := \top^* \cdot \Big(\bigcup_{s \in S_{\langle \mathrm{A} \rangle \psi}} q_s \Big); \quad r_{\langle \overline{\mathrm{A}} \rangle \psi} := \Big(\bigcup_{s \in S_{\langle \overline{\mathrm{A}} \rangle \psi}} q_s \Big) \cdot \top^*.$$

By definition $\mathcal{K}, \rho \models r_{\langle \mathrm{A} \rangle \psi}$ iff $\mathrm{lst}(\rho) \in S_{\langle \mathrm{A} \rangle \psi}$ iff $\mathcal{K}, \rho \models \langle \mathrm{A} \rangle \psi$.

We can now sketch the procedure for "reducing" the MC problem for $\mathsf{A\overline{A}B\overline{B}}$ to the MC problem for $\mathsf{B\overline{B}}$. The idea is to iteratively rewrite a formula Φ of $\mathsf{A\overline{A}B\overline{B}}$ until it gets converted to an (equivalent) formula of $\mathsf{B\overline{B}}$. At each step, we select an occurrence of a subformula of Φ, either of the form $\langle \mathrm{A} \rangle \psi$ or $\langle \overline{\mathrm{A}} \rangle \psi$, devoid of any internal occurrences of modalities $\langle \mathrm{A} \rangle$ and $\langle \overline{\mathrm{A}} \rangle$. For such an occurrence, say $\langle \mathrm{A} \rangle \psi$, we have to compute the set $S_{\langle \mathrm{A} \rangle \psi}$. For this purpose we can run a variant `CheckAux'`$(\mathcal{K}, \Psi, s)$ of the MC procedure `CheckAux`$(\mathcal{K}, \Psi)$, which invokes `Check` at line 2 on the additional parameter (state) s, instead of s_0. For each $s \in S$, we invoke `CheckAux'`$(\mathcal{K}, \neg\psi, s)$, deciding that $s \in S_{\langle \mathrm{A} \rangle \psi}$ iff the procedure returns $\perp$. Then we replace $\langle \mathrm{A} \rangle \psi$ in Φ with a fresh interval property proposition letter associated with the regular expression $r_{\langle \mathrm{A} \rangle \psi}$, obtaining a formula Φ'. To deal with subformulas of the form $\langle \overline{\mathrm{A}} \rangle \psi$, we have to introduce a slight variant of the procedure `Check`, which finds tracks leading to (and not starting from) a given state. Now, if the resulting formula Φ' is in $\mathsf{B\overline{B}}$, then we have finished the conversion process. Otherwise, we can proceed with another iteration of the conversion step over Φ'.

Considering that the sets $S_{\langle \mathrm{A} \rangle \psi}$, $S_{\langle \overline{\mathrm{A}} \rangle \psi'}$ and the regular expressions $r_{\langle \mathrm{A} \rangle \psi}$ and $r_{\langle \overline{\mathrm{A}} \rangle \psi}$ have a size linear in $|S|$, we can conclude with the following theorem.

Theorem 4. *The MC problem for* $\mathsf{A\overline{A}B\overline{B}}$ *formulas over finite Kripke structures is in* $\mathbf{PSPACE}$*.*

By symmetry we can show that MC for $\mathsf{A\overline{A}E\overline{E}}$ formulas is also a $\mathbf{PSPACE}$ problem. The $\mathbf{PSPACE}$-hardness of MC for $\mathsf{B\overline{B}}$ and $\mathsf{A\overline{A}B\overline{B}}$ directly follows from that of the smallest fragment Prop (the purely propositional fragment of HS) which is stated by Theorem 5. As a matter of fact, we prove that Prop is hard for $\mathbf{PSPACE}$ by a reduction from the $\mathbf{PSPACE}$-complete *universality problem for regular expressions* [GJ79] (the problem of deciding, for a regular expression r with $\mathcal{L}(r) \subseteq \Sigma^*$ and $|\Sigma| \geq 2$, whether $\mathcal{L}(r) = \Sigma^*$).

Theorem 5. *The MC problem for formulas of* Prop *over finite Kripke structures is* $\mathbf{PSPACE}$*-hard (under* $\mathbf{LOGSPACE}$ *reductions).*

Proof sketch. Given a regular expression r with $\mathcal{L}(r) \subseteq \Sigma^*$, let us define $\mathcal{K} = (\Sigma, \{s_0\} \cup \Sigma, R, \mu, s_0)$, where $s_0 \notin \Sigma$, $\mu(s_0) = \emptyset$, for $c \in \Sigma$ we have $\mu(c) = \{c\}$, and $R = \{(s_0, c) \mid c \in \Sigma\} \cup \{(c, c') \mid c, c' \in \Sigma\}$. It holds that $\mathcal{L}(r) = \Sigma^* \iff \mathcal{K} \models \top \cdot r$. See Appendix B for the complete proof. $\square$

By Theorems 4 and 5, it immediately follows that model checking formulas of any (proper or improper) sub-fragment of $\mathsf{A\overline{A}B\overline{B}}$ (and of $\mathsf{A\overline{A}E\overline{E}}$) is a $\mathbf{PSPACE}$-complete problem.

VIII. Conclusions

In this paper, we have investigated the MC problem for HS and two large fragments of it, $\mathsf{A\overline{A}B\overline{B}}$ and $\mathsf{A\overline{A}E\overline{E}}$, defining interval labelling via regular expressions. The approach, somehow stemming from [LM16], generalizes both the one of [MMM+16] (which assumes the homogeneity principle) and of [LM13], [LM14] (where labeling is endpoint-based). MC turns out to be non-elementarily decidable and $\mathbf{EXPSPACE}$-hard for full HS (the hardness follows from that of BE under homogeneity [BMM+16a]), and $\mathbf{PSPACE}$-complete for $\mathsf{A\overline{A}B\overline{B}}$, $\mathsf{A\overline{A}E\overline{E}}$, and all their sub-fragments.

Future work will focus on the fragments $\mathsf{A\overline{A}B\overline{B}\overline{E}}$, $\mathsf{A\overline{A}E\overline{B}\overline{E}}$, and $\mathsf{A\overline{A}B\overline{E}}$, which have been proved to be in $\mathbf{EXPSPACE}$ (the first two) and $\mathbf{PSPACE}$-complete (the third one) under the homogeneity assumption [MMP15b], [MMP15a], as well as on the problem of determining the exact complexity of MC for full HS. In addition, we will study the MC problem for HS over *visibly pushdown systems* (VPS), in order to deal with recursive programs and infinite state systems.

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Appendix

A. Proof of Theorem 2

Proof. The proof is by induction on the structure of ψ. The thesis trivially follows for the cases $\psi = r$ (regular expression), $\psi = \neg\psi'$, $\psi = \psi_1 \wedge \psi_2$, and $\psi = \langle \mathrm{B} \rangle\, \psi'$.

Let us now assume $\psi = \langle \overline{\mathrm{B}} \rangle\, \psi'$.

$\texttt{Check}(\mathcal{K}, \psi, s, G, \mathcal{D}(\Phi)) = \top$ if and only if, for some $b'' \in \{1, \ldots, |S| \cdot (2|\psi'| + 1) \cdot 2^{2\sum_{\ell=1}^{u} |r_\ell|} - 1\}$ and some $(G'', \mathcal{D}(\Phi)'', s'') \in \mathsf{Conf}(\mathcal{K}, \psi)$ $(= \mathsf{Conf}(\mathcal{K}, \psi'))$, we have $\texttt{Reach}(\mathcal{K}, \psi', (G, \mathcal{D}(\Phi), s), (G'', \mathcal{D}(\Phi)'', s''), b'') = \top$ and $\texttt{Check}(\mathcal{K}, \psi', s'', G'', \mathcal{D}(\Phi)'') = \top$. We prove first the following claim.

Claim 2. *Let $b \in \mathbb{N}$, $b > 0$. Let $\tilde{\rho} \in \mathrm{Trk}_{\mathcal{K}}$ be a track with $\tilde{s} = \mathrm{lst}(\tilde{\rho})$. Let $\tilde{G}$ be the subset of formulas in $\mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi')$ that hold on some proper prefix of $\tilde{\rho}$. Let $\tilde{\mathcal{D}}(\Phi)$ be the current configuration of states of the DFAs associated with the regular expressions in Φ, reached from the initial states after reading $\mu(\tilde{\rho}(1, |\tilde{\rho}| - 1))$.*

For $(\tilde{G}, \tilde{\mathcal{D}}(\Phi), \tilde{s})$, $(G', \mathcal{D}(\Phi)', s') \in \mathsf{Conf}(\mathcal{K}, \psi')$, $\texttt{Reach}(\mathcal{K}, \psi', (\tilde{G}, \tilde{\mathcal{D}}(\Phi), \tilde{s}), (G', \mathcal{D}(\Phi)', s'), b) = \top$ if and only if there exists $\rho' \in \mathrm{Trk}_{\mathcal{K}}$ such that $\tilde{\rho} \cdot \rho' \in \mathrm{Trk}_{\mathcal{K}}$, $|\rho'| = b$, $\mathrm{lst}(\rho') = s'$, G' is the subset of formulas in $\mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi')$ that hold on some proper prefix of $\tilde{\rho} \cdot \rho'$, and $\mathcal{D}(\Phi)'$ is the current configuration of the DFAs associated with the regular expressions of Φ, after reading $\mu(\tilde{\rho} \cdot \rho'(1, |\tilde{\rho} \cdot \rho'| - 1))$.

Proof. The proof is by induction on $b \geq 1$.

If $b = 1$, $\texttt{Reach}(\mathcal{K}, \psi', (\tilde{G}, \tilde{\mathcal{D}}(\Phi), \tilde{s}), (G', \mathcal{D}(\Phi)', s'), b) = \top$ iff $\texttt{Compatible}(\mathcal{K}, \psi', (\tilde{G}, \tilde{\mathcal{D}}(\Phi), \tilde{s}), (G', \mathcal{D}(\Phi)', s')) = \top$. This happens if and only if:

1. $(\tilde{s}, s') \in R$—i.e., $(\tilde{s}, s')$ is an edge of $\mathcal{K}$;
2. $\texttt{advance}(\tilde{\mathcal{D}}(\Phi), \mu(\tilde{s})) = \mathcal{D}(\Phi)'$;
3. $\tilde{G} \subseteq G'$;
4. for each $\varphi \in (G' \setminus \tilde{G})$, $\texttt{Check}(\mathcal{K}, \varphi, \tilde{s}, \tilde{G} \cap \mathsf{Subf}_{\langle \mathrm{B} \rangle}(\varphi), \tilde{\mathcal{D}}(\Phi)) = \top$;
5. for each $\varphi \in (\mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi') \setminus G')$, $\texttt{Check}(\mathcal{K}, \varphi, \tilde{s}, \tilde{G} \cap \mathsf{Subf}_{\langle \mathrm{B} \rangle}(\varphi), \tilde{\mathcal{D}}(\Phi)) = \bot$.

Let $\rho' = s'$. ($\Rightarrow$) By the inductive hypothesis (of the external theorem over $\tilde{\rho}$), by 4. it follows that $\mathcal{K}, \tilde{\rho} \models \varphi$ for each $\varphi \in (G' \setminus \tilde{G})$. By 5. it follows that $\mathcal{K}, \tilde{\rho} \not\models \varphi$ for each $\varphi \in (\mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi') \setminus G')$. The claim follows.

Conversely, ($\Leftarrow$) 1., 2., and 3. easily follow. Moreover it must hold that $\mathcal{K}, \tilde{\rho} \models \varphi$ for each $\varphi \in (G' \setminus \tilde{G})$, and $\mathcal{K}, \tilde{\rho} \not\models \varphi$ for each $\varphi \in (\mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi') \setminus G')$: 4. and 5. follow by the inductive hypothesis (of the external theorem).

If $b \geq 2$, $\texttt{Reach}(\mathcal{K}, \psi', (\tilde{G}, \tilde{\mathcal{D}}(\Phi), \tilde{s}), (G', \mathcal{D}(\Phi)', s'), b) = \top$ if and only if, for some $(G_3, \mathcal{D}(\Phi)_3, s_3) \in \mathsf{Conf}(\mathcal{K}, \psi')$, $\texttt{Reach}(\mathcal{K}, \psi', (\tilde{G}, \tilde{\mathcal{D}}(\Phi), \tilde{s}), (G_3, \mathcal{D}(\Phi)_3, s_3), \lfloor b/2 \rfloor) = \top$ and $\texttt{Reach}(\mathcal{K}, \psi', (G_3, \mathcal{D}(\Phi)_3, s_3), (G', \mathcal{D}(\Phi)', s'), b - \lfloor b/2 \rfloor) = \top$.

($\Rightarrow$) By the inductive hypothesis (over b), there exists $\rho_3 \in \mathrm{Trk}_{\mathcal{K}}$ such that $\tilde{\rho} \cdot \rho_3 \in \mathrm{Trk}_{\mathcal{K}}$, $|\rho_3| = \lfloor b/2 \rfloor$, $\mathrm{lst}(\rho_3) = s_3$, G_3 is the subset of subformulas in $\mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi')$ that hold on some proper prefix of $\tilde{\rho} \cdot \rho_3$, and $\mathcal{D}(\Phi)_3$ is the current configuration of the DFAs associated with the regular expressions in Φ, after reading $\mu(\tilde{\rho} \cdot \rho_3(1, |\tilde{\rho} \cdot \rho_3| - 1))$.

By the inductive hypothesis (over b, applied to the track $\tilde{\rho} \cdot \rho_3$), there exists $\rho' \in \mathrm{Trk}_{\mathcal{K}}$ such that $\tilde{\rho} \cdot \rho_3 \cdot \rho' \in \mathrm{Trk}_{\mathcal{K}}$, $|\rho'| = b - \lfloor b/2 \rfloor$, $\mathrm{lst}(\rho') = s'$, G' is the subset of subformulas in $\mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi')$ that hold on some proper prefix of $\tilde{\rho} \cdot \rho_3 \cdot \rho'$, and $\mathcal{D}(\Phi)'$ is the current configuration of the DFAs associated with the regular expressions in Φ, after reading $\mu(\tilde{\rho} \cdot \rho_3 \cdot \rho'(1, |\tilde{\rho} \cdot \rho_3 \cdot \rho'| - 1))$. The claim follows, as $\rho_3 \cdot \rho' \in \mathrm{Trk}_{\mathcal{K}}$ and $|\rho_3 \cdot \rho'| = b$.

($\Leftarrow$) Conversely, there exists $\rho' \in \mathrm{Trk}_{\mathcal{K}}$ such that $\tilde{\rho} \cdot \rho' \in \mathrm{Trk}_{\mathcal{K}}$, $|\rho'| = b \geq 2$, $\mathrm{lst}(\rho') = s'$, G' is the subset of subformulas in $\mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi')$ that hold on some proper prefix of $\tilde{\rho} \cdot \rho'$, and $\mathcal{D}(\Phi)'$ is the current configuration of the DFAs associated with the regular expressions in Φ, after reading $\mu(\tilde{\rho} \cdot \rho'(1, |\tilde{\rho} \cdot \rho'| - 1))$. Let us split $\rho' = \rho_3 \cdot \rho_4$, where $|\rho_3| = \lfloor b/2 \rfloor$ and $|\rho_4| = b - \lfloor b/2 \rfloor$. Let $(G_3, \mathcal{D}(\Phi)_3, s_3) \in \mathsf{Conf}(\mathcal{K}, \psi')$ be such that $\mathcal{D}(\Phi)_3$ is the current configuration of the DFAs associated with the regular expressions in Φ, after reading $\mu(\tilde{\rho} \cdot \rho_3(1, |\tilde{\rho} \cdot \rho_3| - 1))$, $s_3 = \mathrm{lst}(\rho_3)$, G_3 is the subset of subformulas in $\mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi')$ that hold on some proper prefix of $\tilde{\rho} \cdot \rho_3$. By the inductive hypothesis (on b over $\tilde{\rho} \cdot \rho_3$), $\texttt{Reach}(\mathcal{K}, \psi', (G_3, \mathcal{D}(\Phi)_3, s_3), (G', \mathcal{D}(\Phi)', s'), b - \lfloor b/2 \rfloor) = \top$. Moreover, by the inductive hypothesis (on b over $\tilde{\rho}$), $\texttt{Reach}(\mathcal{K}, \psi', (\tilde{G}, \tilde{\mathcal{D}}(\Phi), \tilde{s}), (G_3, \mathcal{D}(\Phi)_3, s_3), \lfloor b/2 \rfloor) = \top$.

Hence both the recursive calls at line 6 return $\top$, when at line 5 $(G_3, \mathcal{D}(\Phi)_3, s_3)$ is considered by the loop. Thus $\texttt{Reach}(\mathcal{K}, \psi', (\tilde{G}, \tilde{\mathcal{D}}(\Phi), \tilde{s}), (G', \mathcal{D}(\Phi)', s'), b)$ returns $\top$.

This concludes the proof of the claim. $\square$

($\Rightarrow$) Let us now assume that in $\texttt{Check}$, at lines 15–19, for some $b'' \in \{1, \ldots, |S| \cdot (2|\psi'| + 1) \cdot 2^{2\sum_{\ell=1}^{u} |r_\ell|} - 1\}$ and some $(G'', \mathcal{D}(\Phi)'', s'') \in \mathsf{Conf}(\mathcal{K}, \psi)$ $(= \mathsf{Conf}(\mathcal{K}, \psi'))$, we have $\texttt{Reach}(\mathcal{K}, \psi', (G, \mathcal{D}(\Phi), s), (G'', \mathcal{D}(\Phi)'', s''), b'') = \top$ and $\texttt{Check}(\mathcal{K}, \psi', s'', G'', \mathcal{D}(\Phi)'') = \top$. By the previous claim, there exists $\rho'' \in \mathrm{Trk}_{\mathcal{K}}$ such that $\rho \cdot \rho'' \in \mathrm{Trk}_{\mathcal{K}}$, $\mathrm{lst}(\rho'') = s''$, G'' is the subset of subformulas in $\mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi')$ that hold on some proper prefix of $\rho \cdot \rho''$, and $\mathcal{D}(\Phi)''$ is the current configuration of the DFAs associated with the regular expressions of Φ, after reading $\mu(\rho \cdot \rho''(1, |\rho \cdot \rho''| - 1))$. By the inductive hypothesis, since $\texttt{Check}(\mathcal{K}, \psi', s'', G'', \mathcal{D}(\Phi)'') = \top$, we have $\mathcal{K}, \rho \cdot \rho'' \models \psi'$. Thus $\mathcal{K}, \rho \models \langle \overline{\mathrm{B}} \rangle\, \psi'$.

($\Leftarrow$) Conversely, if $\mathcal{K}, \rho \models \langle \overline{\mathrm{B}} \rangle\, \psi'$, we have $\mathcal{K}, \rho \cdot \rho'' \models \psi'$ for some $\rho'' \in \mathrm{Trk}_{\mathcal{K}}$, with $\rho \cdot \rho'' \in \mathrm{Trk}_{\mathcal{K}}$. By the exponential small-model Theorem 1, there exists $\rho' \in \mathrm{Trk}_{\mathcal{K}}$ such that $\mathrm{lst}(\rho'') = \mathrm{lst}(\rho')$, $|\rho'| \leq |S| \cdot (2|\psi'| + 1) \cdot 2^{2\sum_{\ell=1}^{u} |r_\ell|} - 1$ (the factor 2 in front of $|\psi'|$ is due to the fact that the exponential small-model property requires a formula in NNF), $\rho \cdot \rho' \in \mathrm{Trk}_{\mathcal{K}}$ and $\mathcal{K}, \rho \cdot \rho' \models \psi'$. Let G' be the subset of subformulas in $\mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi') = \mathsf{Subf}_{\langle \mathrm{B} \rangle}(\psi)$ that hold on some proper prefix of $\rho \cdot \rho'$, and $\mathcal{D}(\Phi)'$ be the current configuration of the DFAs associated with the regular expressions in Φ, after reading $\mu(\rho \cdot \rho'(1, |\rho \cdot \rho'| - 1))$. By the inductive hypothesis (over $\rho \cdot \rho'$), $\texttt{Check}(\mathcal{K}, \psi', \mathrm{lst}(\rho'), G', \mathcal{D}(\Phi)') = \top$. By the previous claim, $\texttt{Reach}(\mathcal{K}, \psi', (G, \mathcal{D}(\Phi), s), (G', \mathcal{D}(\Phi)', \mathrm{lst}(\rho')), |\rho'|) = \top$, hence $\texttt{Check}(\mathcal{K}, \psi, s, G, \mathcal{D}(\Phi)) = \top$.

This concludes the proof of the theorem. $\square$

B. Proof of Theorem 5

Proof. Given a regular expression r with $\mathcal{L}(r) \subseteq \Sigma^*$, let us define $\mathcal{K} = (\Sigma, \{s_0\} \cup \Sigma, R, \mu, s_0)$, where $s_0 \notin \Sigma$, $\mu(s_0) = \emptyset$, for $c \in \Sigma$ we have $\mu(c) = \{c\}$, and $R = \{(s_0, c) \mid c \in \Sigma\} \cup \{(c, c') \mid c, c' \in \Sigma\}$. It holds that

$$\mathcal{L}(r) = \Sigma^* \iff \mathcal{K} \models \top \cdot \overline{r},$$

where $\overline{r}$ is a RE over Σ, *syntactically* the same as r. Note that whereas r is a standard regular expression—defining a finitary language over Σ—$\overline{r}$, even though syntactically the same as r, defines a finitary language *over* 2^{Σ}, as pointed out in Section II. The distinction between r and $\overline{r}$ is kept in the rest of the proof in order to avoid confusion between the two "roles" of r.

We show by induction on the structure of r that, for all $w \in \Sigma^*$, $w \in \mathcal{L}(r) \iff \mathcal{K}, w \models \overline{r}$. The thesis follows as $\mathcal{K}, w \models \overline{r}$ if and only if $\mathcal{K}, s_0 \cdot w \models \top \cdot \overline{r}$.

- $r = \varepsilon$. Then, $w \in \mathcal{L}(\varepsilon)$ iff $w = \varepsilon$ iff $\mu(w) \in \mathcal{L}(\overline{\varepsilon}) = \{\varepsilon\}$ iff $\mathcal{K}, w \models \overline{\varepsilon}$.
- $r = c \in \Sigma$. Then, we have $w \in \mathcal{L}(c)$ iff $w = c$, thus $\mu(w) = \{c\} \in \mathcal{L}(\overline{c})$, and $\mathcal{K}, w \models \overline{c}$. Conversely, if $\mathcal{K}, w \models \overline{c}$ we have $\mu(w) \in \mathcal{L}(\overline{c}) = \{A \in 2^{\Sigma} \mid c \in A\}$. In particular $|w| = 1$. Moreover, by definition of μ, $\mu(w)$ is a singleton, hence $\mu(w) = \{c\}$. By definition of $\mathcal{K}$, $w = c$, thus $w \in \mathcal{L}(c)$.
- $r = r_1 \cdot r_2$. $w \in \mathcal{L}(r_1 \cdot r_2)$ iff $w = w_1 \cdot w_2$ and $w_1 \in \mathcal{L}(r_1)$ and $w_2 \in \mathcal{L}(r_2)$. By applying the inductive hypothesis, $\mathcal{K}, w_1 \models \overline{r_1}$ and $\mathcal{K}, w_2 \models \overline{r_2}$, thus $\mu(w_1) \in \mathcal{L}(\overline{r_1})$ and $\mu(w_2) \in \mathcal{L}(\overline{r_2})$. It follows that $\mu(w) = \mu(w_1) \cdot \mu(w_2) \in \mathcal{L}(\overline{r_1}) \cdot \mathcal{L}(\overline{r_2}) = \mathcal{L}(\overline{r_1 \cdot r_2})$, namely $\mathcal{K}, w \models \overline{r_1 \cdot r_2}$. Conversely, $\mu(w) \in \mathcal{L}(\overline{r_1 \cdot r_2}) = \mathcal{L}(\overline{r_1}) \cdot \mathcal{L}(\overline{r_2})$. Hence $\mu(w_1) \in \mathcal{L}(\overline{r_1})$ and $\mu(w_2) \in \mathcal{L}(\overline{r_2})$, for some $w_1 \cdot w_2 = w$. By the inductive hypothesis, $w_1 \in \mathcal{L}(r_1)$ and $w_2 \in \mathcal{L}(r_2)$, hence $w \in \mathcal{L}(r_1 \cdot r_2)$.
- $r = r_1 \cup r_2$. $w \in \mathcal{L}(r_1 \cup r_2)$ iff $w \in \mathcal{L}(r_i)$ for some $i = 1, 2$. By the inductive hypothesis this is true iff $\mathcal{K}, w \models \overline{r_i}$, iff $\mu(w) \in \mathcal{L}(\overline{r_i})$, iff $\mu(w) \in \mathcal{L}(\overline{r_1 \cup r_2})$, iff $\mathcal{K}, w \models \overline{r_1 \cup r_2}$.
- $r = r_1^*$. The thesis trivially holds if $w = \varepsilon$. Let us now assume $w \neq \varepsilon$. $w \in \mathcal{L}(r_1^*)$ iff $w = w_1 \cdots w_t$, $t \geq 1$, such that $w_\ell \in \mathcal{L}(r_1)$ for all $1 \leq \ell \leq t$. By the inductive hypothesis, $\mathcal{K}, w_\ell \models \overline{r_1}$, thus $\mu(w_\ell) \in \mathcal{L}(\overline{r_1})$, and $\mu(w) \in \mathcal{L}(\overline{r_1^*})$. We conclude that $\mathcal{K}, w \models \overline{r_1^*}$. Conversely, $\mu(w) \in \mathcal{L}(\overline{r_1^*}) = (\mathcal{L}(\overline{r_1}))^*$, hence it must be the case that $w = w_1 \cdots w_t$, $t \geq 1$, such that $\mu(w_\ell) \in \mathcal{L}(\overline{r_1})$. By the inductive hypothesis, $w_\ell \in \mathcal{L}(r_1)$, hence $w \in \mathcal{L}(r_1^*)$.

Finally, we can build $\mathcal{K}$ using logarithmic working space. $\square$