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PREPRINT

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Preprint nr.: 2/2012

Reports available from: http://www.dimi.uniud.it/preprints

Scale function vs Topological entropy

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25 settembre 2012

Abstract

In the realm of topological automorphisms of totally disconnected locally compact groups, the scale function is compared with the topological entropy. Indeed the logarithm of the scale function is proved to be always smaller than the topological entropy; moreover, an example shows that this inequality can be strict, and on the other hand conditions are given to force these two invariants to coincide. Finally, several properties of the scale function inspired by those of the topological entropy are presented.

Key words: scale function, topological entropy, totally disconnected locally compact group, automorphism
2010 AMS Subject Classification: 37B40, 22D05, 22D40, 54H11, 54H20, 54C70.

1 Introduction

The scale function was introduced by Willis in [15] for topological automorphisms of totally disconnected locally compact groups and developed in his later works; among them we mention [16, 17, 18].

On the other hand, Adler, Konheim and McAndrew introduced in [1] the topological entropy for continuous selfmaps of compact spaces, while later on Bowen in [3] gave a different definition of topological entropy for uniformly continuous selfmaps of metric spaces, and this was extended to uniformly continuous selfmaps of uniform spaces by Hood in [10]. As explained in detail in [7], this definition can be simplified in the case of continuous endomorphisms of totally disconnected locally compact groups.

In particular, for a topological automorphism of a totally disconnected locally compact group, the logarithm of the scale function and the topological entropy seem to be strongly related. A question in this direction was posed by T. Weigel, who asked for a possible relation of the scale function with either the topological entropy or the algebraic entropy. Even if they do not coincide in general, we see in this paper that the values of the logarithm of the scale function and of the topological entropy can be obtained in a similar way, and this permits to find the precise relation between them.

Further aspects of the connection of the scale function to the topological and the algebraic entropy are discussed in the forthcoming paper [2].

So in this paper we are mainly concerned with a totally disconnected locally compact group G and a topological automorphism $\phi : G \to G$; when not explicitly said, we are assuming to be under these hypotheses. It is worth recalling immediately that a totally disconnected locally compact group G has as a local base at e_G the family $\mathcal{B}(G)$ of all open compact subgroups of G, as proved by van Dantzig in [14].

We start now giving the precise definition of scale function as it was introduced by Willis in [18]. For G a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism, the *scale* of ϕ is

$$s_G(\phi) = \min\{s_G(\phi, U) : U \in \mathcal{B}(G)\}, \tag{1.1}$$

where

$$s_G(\phi, U) = [\phi(U) : U \cap \phi(U)].$$

Note that for every $U \in \mathcal{B}(G)$ the index $s_G(\phi, U)$ is finite as $U \cap \phi(U)$ is open and $\phi(U)$ is compact. We use the notations $s(\phi, U)$ and $s(\phi)$ when the group G is clear from the context.

Since the scale function is defined as a minimum there exists $U \in \mathcal{B}(G)$ for which this minimum realizes, that is $s(\phi) = s(\phi, U)$; such U is called *minimizing* for ϕ [17].

Moreover, we say that a subgroup $U \in \mathcal{B}(G)$ is *ϕ-invariant* if $\phi(U) \subseteq U$, *inversely ϕ-invariant* if $U \subseteq \phi(U)$ (i.e., $\phi^{-1}(U) \subseteq U$), and *ϕ-stable* if $\phi(U) = U$ (i.e., U is both ϕ-invariant and inversely ϕ-invariant).

Clearly, if U is ϕ-invariant, then U is minimizing for ϕ and $s(\phi) = 1$. Also the converse implication holds true, in the sense that $s(\phi) = 1$ if and only if there exists a ϕ-invariant $U \in \mathcal{B}(G)$. This is the case for example when G is either compact or discrete.

Since U is minimizing for ϕ precisely when U is minimizing for ϕ^{-1}, also the inversely ϕ-invariant $U \in \mathcal{B}(G)$ are minimizing for ϕ (see Lemma 2.1).

On the other hand, if one has to use only the definition of the scale function, minimizing subgroups that are not ϕ-invariant or inversely ϕ-invariant become quite hard to come by, since in (1.1) one has to check all subgroups from the large filter base $\mathcal{B}(G)$. So, in order to characterize and find minimizing subgroups, a different approach is adopted by Willis and we describe it in what follows. For $U \in \mathcal{B}(G)$ let

$$U_{\phi+} = \bigcap_{n\in\mathbb{N}} \phi^n(U) \quad \text{and} \quad U_{\phi-} = \bigcap_{n\in\mathbb{N}} \phi^{-n}(U); \tag{1.2}$$

and also

$$U_{\phi++} = \bigcup_{n\in\mathbb{N}} \phi^n(U_{\phi+}) \quad \text{and} \quad U_{\phi--} = \bigcup_{n\in\mathbb{N}} \phi^{-n}(U_{\phi-}). \tag{1.3}$$

Note that $U_{\phi-} = U_{\phi^{-1}+}$ and see the diagram in (2.1) below. When the automorphism ϕ is clear from the context it is omitted from these notations.

So U is:

(a) *tidy above* for ϕ if $U = U_+U_-$ (i.e., $U = U_-U_+$);

(b) *tidy below* for ϕ if U_{++} is closed;

(c) *tidy* for ϕ if it is tidy above and tidy below for ϕ.

The consequence of the so-called "tidying procedure" given by Willis is the following fundamental theorem showing that the minimizing subgroups are precisely the tidy subgroups.

Theorem 1.1. [17, Theorem 3.1] *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and $U \in \mathcal{B}(G)$. Then U is minimizing for ϕ if and only if U is tidy for ϕ. In this case*

$$s(\phi) = [\phi(U_+) : U_+].$$

Note that the index $[\phi(U_+) : U_+]$ is finite as $U_+ = U \cap \phi(U_+)$ and U is open, while $\phi(U_+)$ is compact.

We recall now the definition of topological entropy in this setting following [7]. Let G be a totally disconnected locally compact group, $\phi : G \to G$ a continuous endomorphism and $U \in \mathcal{B}(G)$. For an integer $n \geq 0$ let

$$U_n = \bigcap_{k=0}^{n} \phi^k(U) \quad \text{and} \quad U_{-n} = \bigcap_{k=-n}^{0} \phi^k(U). \tag{1.4}$$

The *topological entropy of ϕ with respect to U* is given by the following limit, which is proved to exist,

$$H_{top}(\phi, U) = \lim_{n\to\infty} \frac{\log[U : U_{-n}]}{n};$$

and the *topological entropy* of ϕ is

$$h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{B}(G)\}.$$

The following limit free formula for the computation of the topological entropy proved in [8] gives the possibility to easily compare the scale function with the topological entropy. An analogous formula for the topological entropy of continuous endomorphisms of totally disconnected compact groups was previously given in [5].

Theorem 1.2. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and $U \in \mathcal{B}(G)$. Then*

$$H_{top}(\phi, U) = \log[\phi(U_+) : U_+].$$

In [8] this formula is applied to verify the basic properties of the topological entropy, well-known for compact groups, also for topological automorphisms of totally disconnected locally compact groups. These properties are the so-called Invariance under conjugation, Logarithmic Law, Monotonicity for subgroups and quotients, Weak Addition Theorem and Continuity for inverse limits (see Fact 4.1 below).

Contents of the paper. The paper is organized as follows.

In the first part of Section 2 we recall some basic properties of the tidy subgroups, which are applied in the following sections to prove the main result of this paper. Then we consider a subgroup introduced by Willis, which is strictly related to the scale functions; this subgroup is the nub, defined as the intersection of all tidy subgroups.

In Section 3 we compare the values of the logarithm of the scale function with those of the topological entropy. Indeed, we see that for G a locally compact group and $\phi : G \to G$ a topological automorphism, they are respectively the min and the sup of the same subset of $\log \mathbb{N}_+$, that is

$$\{\log[\phi(U_+) : U_+] : U \in \mathcal{B}(G)\}.$$

An immediate consequence is that the inequality

$$\log s(\phi) \leq h_{top}(\phi) \tag{1.5}$$

holds in general. Moreover, in Example 3.3 we see that this inequality can be strict, even in the abelian case. On the other hand, equality holds in (1.5) when the tidy subgroups form a local base at e_G (see Proposition 3.5 and Theorem 3.7).

In Section 4 we give the properties of the scale function with respect to the typical properties of the topological entropy. Invariance under conjugation, Logarithmic Law and Monotonicity for subgroups and quotients were proved by Willis, while we see that also Weak Addition Theorem holds true and we discuss Continuity for direct and inverse limits.

In Section 5 we give an explicit computation of the scale function of the topological automorphisms of $\mathbb{Q}_p^n$, where $\mathbb{Q}_p$ denotes the field of p-adic numbers (see Theorem 5.3). This result is inspired by the so-called p-adic Yuzvinski Formula for the topological entropy. Indeed, the so-called Yuzvinski Formula was proved in [20] by Yuzvinski; it gives the values of the topological entropy of topological automorphisms ϕ of $\widehat{\mathbb{Q}}^n$ in terms of the Mahler measure of the characteristic polynomial of ϕ. A different and clear proof of the Yuzvinski Formula is given in [11], it is based on the computation of the topological entropy of topological automorphisms of $\mathbb{Q}_p^n$, given by the result that we call p-adic Yuzvinski Formula.

In the second part of Section 5 we assume the locally compact totally disconnected group G to be abelian. Under the hypothesis that G is covered by its compact subgroups, that ensures total disconnectedness of the Pontryagin dual $\widehat{G}$ of G, we prove that

$$s(\phi) = s(\widehat{\phi}),$$

where $\widehat{\phi} : \widehat{G} \to \widehat{G}$ is the dual automorphism of the topological automorphism ϕ of G (see Theorem 5.7). This is a so-called Bridge Theorem, inspired by the analogous one connecting the topological entropy with the algebraic entropy in the same setting (see [6]).

Notation and terminology

As usual, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}_+$ the set of positive integers, $\mathbb{P}$ denotes the set of all prime numbers, $\mathbb{Z}$ denotes the group of integers and $\mathbb{T}$ denotes the circle group with its usual topology. For $p \in \mathbb{P}$, $\mathbb{Z}(p) = \mathbb{Z}/p\mathbb{Z}$ denotes the cyclic group of order p, $\mathbb{J}_p$ denotes the group/ring of p-adic integers and $\mathbb{Q}_p$ denotes the field of p-adic numbers.

Let G be a topological abelian group, then the Pontryagin dual $\widehat{G}$ of G is the (abelian) group of all continuous homomorphisms $\chi\colon G \to \mathbb{T}$ (i.e., characters), endowed with the compact-open topology. If $\phi\colon G \to G$ is a continuous endomorphism, then its dual homomorphism $\widehat{\phi}\colon \widehat{G} \to \widehat{G}$ is defined by $\widehat{\phi}(\chi) = \chi \circ \phi$ for every $\chi \in \widehat{G}$. If G is a locally compact abelian group, so is its dual group $\widehat{G}$, and the dual endomorphism $\widehat{\phi} : \widehat{G} \to \widehat{G}$ is continuous.

2 Scale function and tidy subgroups

In the first part of this section we are mainly concerned with basic observations on tidy subgroups.

The next lemma collects in particular known immediate examples of minimizing subgroups.

Lemma 2.1. *Let G be a totally disconnected locally compact group, $\phi\colon G \to G$ a topological automorphism and $U \in \mathcal{B}(G)$. Then:*

(a) if U is ϕ-invariant, then U is minimizing for ϕ and $s(\phi) = s(\phi, U) = 1$;

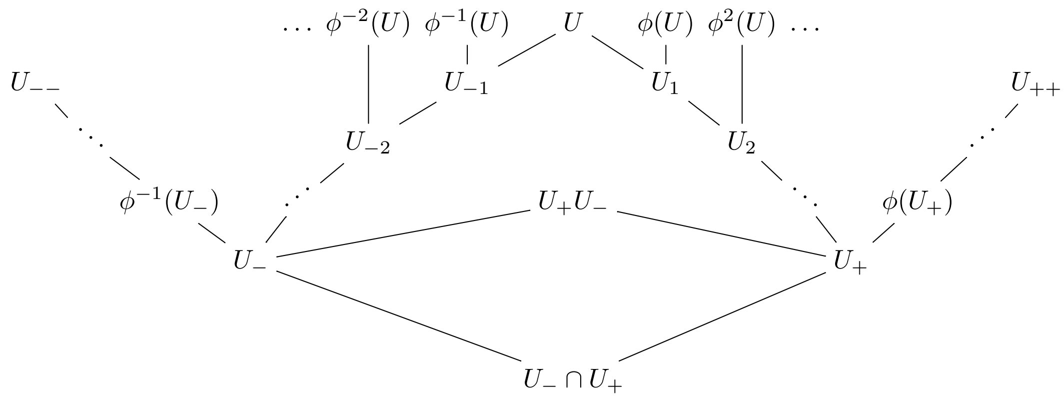
(b) U is minimizing for ϕ if and only if U is minimizing for ϕ^{-1};

(c) if U is inversely ϕ-invariant, then U is minimizing for ϕ;

(d) consequently, $s(\phi) = 1$ if and only if there exists a ϕ-invariant $U \in \mathcal{B}(G)$.

Proof. (a) follows immediately from the definition of scale function, (b) is [15, Corollary 1], (c) follows from (a) and (b), while (d) from (a) and the definition of scale function. $\square$

As recalled in the Introduction, if G is a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism, for a subgroup $U \in \mathcal{B}(G)$ Willis had the nice idea to consider the following diagram in the lattice of all subgroups of G (the subgroups appearing in the diagram are defined in (1.2), (1.3) and (1.4) above):



(2.1)

The motivation to introduce these subgroups is to measure the extent to which the subgroup U is ϕ-invariant or inversely ϕ-invariant. Indeed, U is ϕ-invariant if and only if $U_- = U$, while U is inversely ϕ-invariant if and only if $U_+ = U$.

The subgroup U_+ is compact and it is the largest inversely ϕ-invariant subgroup contained in U; moreover, we have an increasing chain of subgroups

$$U_+ \subseteq \phi(U_+) \subseteq \ldots \subseteq \phi^n(U_+) \subseteq \ldots \subseteq U_{++} = \bigcup_{n\in\mathbb{N}} \phi^n(U_+),$$

where all indices $[\phi^{n+1}(U_+) : \phi^n(U_+)]$ are finite, as $[\phi(U_+) : U_+]$ is finite. Hence U_{++}, which is the increasing union of this chain, is a subgroup of G that contains U_+. If U_{++} is closed, then it is locally compact, hence a Baire space; so there exists an integer $n \geq 0$ such that $\phi^n(U_+)$ is open in U_{++}; in this case U_+ is also open in U_{++}.

The next Lemma 2.3 was proved in [15, Lemma 1] in the case of inner automorphism; it provides characterizations of the tidy above subgroups. The next elementary fact from group theory is needed.

Claim 2.2. *Let G be a group and let A, B, C be subgroups of G. If $C \subseteq B$ and $B \subseteq A \cdot C$, then $B = (A \cap B)C = C(A \cap B)$.*

Lemma 2.3. *Let G be a locally compact totally disconnected group, $\phi\colon G \to G$ a topological automorphism and $U \in \mathcal{B}(G)$. Then the following conditions are equivalent:*

(a) U is tidy above for ϕ;

(b) $\phi(U) = \phi(U_+)(U \cap \phi(U))$;

(c) $\phi^n(U) = \phi^n(U_+)U_n$ for every integer $n \geq 0$;

(d) $U_+ \cap uU_- \neq \emptyset$ for every $u \in U$.

Proof. (a)⇒(b) Let $U \in \mathcal{B}(G)$ be tidy above for ϕ. This means that $U = U_-U_+$ and so $\phi(U) = \phi(U_-)\phi(U_+)$. Moreover, $\phi(U_-) \subseteq U_- \subseteq U$ and then $\phi(U) \subseteq U \cdot \phi(U_+)$. Now Claim 2.2 applied to U, $\phi(U)$ and $\phi(U_+)$ yields $\phi(U) = \phi(U_+)(U \cap \phi(U))$.

(b)⇒(c) Let $n \geq 0$. The inclusion $U_n \subseteq U$ is always satisfied, so

$$\phi(U_n) \subseteq \phi(U) = \phi(U_+)(U \cap \phi(U)) \subseteq \phi(U_+)U,$$

thus Claim 2.2 applied to U, $\phi(U_n)$ and $\phi(U_+)$ yields

$$\phi(U_n) = \phi(U_+)(U \cap \phi(U_n)) = \phi(U_+)U_{n+1}. \tag{2.2}$$

Using (2.2) we prove by induction the condition in (c). Indeed, the case $n = 0$ is clear and the case $n = 1$ is exactly the condition in (b). Now assume that $\phi^n(U) = \phi^n(U_+)U_n$. Therefore $\phi^{n+1}(U) = \phi^{n+1}(U_+)\phi(U_n) = \phi^{n+1}(U_+)U_{n+1}$, where the last equality follows from (2.2) noting also that $\phi(U_+) \subseteq \phi^{n+1}(U_+)$.

(c)⇒(d) Let $u \in U$ and consider, for every integer $n \geq 0$, the subset $C_n(u) = U_+ \cap uU_{-n}$. These subsets are compact and satisfy $C_{n+1}(u) \subseteq C_n(u)$. Moreover, since $\phi^n(U_{-n}) = U_n$,

$$\begin{aligned} C_n(u) &= \{z \in U_+ : z \in uU_{-n}\} \\ &= \{z \in U_+ : u^{-1} \in z^{-1}U_{-n}\} \\ &= \{z \in U_+ : \phi^n(u^{-1}) \in \phi^n(z^{-1})U_n\}. \end{aligned}$$

Then $C_n(u)$ is non-empty in view of the condition in (c). By the compactness of U_+, the intersection $C = \bigcap_{n\in\mathbb{N}} C_n(u)$ is non-empty. Moreover, it coincides with $U_+ \cap uU_-$; in fact, the inclusion $U_+ \cap uU_- \subseteq C$ is clear. To verify the converse inclusion let $z \in C$, that exists since C is non-empty; then $z \in U_+ \cap uU_{-n}$ for every $n \geq 0$, in particular $z \in U_+$ and $u^{-1}z \in U_{-n}$ for every $n \geq 0$, that is $z \in U_+ \cap uU_-$.

(d)⇒(a) For every $u \in U$ there exist $u_+ \in U_+$ and $u_- \in U_-$ such that $u_+ = uu_-$, that is $u = u_+(u_-)^{-1}$. This means that $U \subseteq U_+U_-$, that is U is tidy above for ϕ. □

This lemma has important consequences. In particular, the next corollary of Lemma 2.3 and Theorem 1.1 is one of the two main ingredients to prove in the next section the inequality announced in (1.5).

Corollary 2.4. *Let G be a locally compact totally disconnected group, $\phi\colon G \to G$ a topological automorphism and $U \in \mathcal{B}(G)$. Then $s(\phi, U) \geq [\phi(U_+) : U_+]$; equality holds exactly when U is tidy above for ϕ.*

In particular,

$$s(\phi) = \min\{[\phi(U_+) : U_+] : U \in \mathcal{B}(G)\}.$$

Proof. Since $\phi(U) \supseteq \phi(U_+)U_1$, we have

$$s(\phi, U) = [\phi(U) : U_1] \geq [\phi(U_+)U_1 : U_1].$$

Moreover,

$$[\phi(U_+)U_1 : U_1] = [\phi(U_+) : U_1 \cap \phi(U_+)] = [\phi(U_+) : U_+].$$

This proves that $s(\phi, U) \geq [\phi(U_+) : U_+]$.

If U is tidy above, then $\phi(U) = \phi(U_+)U_1$ by Lemma 2.3, and hence we have the equality $s(\phi, U) = [\phi(U_+) : U_+]$.

From what we have just proved it follows that $s(\phi) \leq \min\{[\phi(U_+) : U_+] : U \in \mathcal{B}(G)\}$. Equality holds, since Theorem 1.1 yields that $s(\phi) = s(\phi, V) = [\phi(V_+) : V_+]$ for some $V \in \mathcal{B}(G)$ tidy for ϕ. □

Another consequence of Lemma 2.3 is the following result.

Corollary 2.5. *Let G be a locally compact totally disconnected group, $\phi\colon G \to G$ a topological automorphism and $U \in \mathcal{B}(G)$. There exists an integer $n \geq 0$ such that U_n is tidy above for ϕ.*

Proof. Consider the subfamily $\{\phi(U_n)\}_{n\geq 0}$ of $\mathcal{B}(G)$, and note that $\phi(U_n) \supseteq \phi(U_{n+1})$ for every $n \geq 0$, and $\phi(U_+) = \bigcap_{n\in\mathbb{N}} \phi(U_n)$. Consider the set $\phi(U_+)U$, which is a compact and open neighborhood of $\phi(U_+)$. There exists an integer $n \geq 0$ such that $\phi(U_n) \subseteq \phi(U_+)U$. Apply now Claim 2.2 to U, $\phi(U_n)$ and $\phi(U_+)$ to obtain

$$\phi(U_n) = \phi(U_+)(U \cap \phi(U_n)) = \phi(U_+)U_{n+1}.$$

Since $U_+ = (U_n)_+$ and $U_{n+1} = U_n \cap \phi(U_n)$ we have that

$$\phi(U_n) = \phi((U_n)_+)(U_n \cap \phi(U_n)).$$

In view of Lemma 2.3 this means that U_n is tidy above for ϕ. □

In the second part of this section we consider a subgroup defined in [19], which is strongly related to tidy subgroups and the scale function. For G a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism, the *nub* of ϕ is

$$\mathrm{nub}(\phi) = \bigcap\{U \in \mathcal{B}(G) : U \text{ is tidy for } \phi\}.$$

The following useful characterization of $\mathrm{nub}(\phi)$ should be mentioned:

Fact 2.6. [19, Corollary 4.7] *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. Then* $\mathrm{nub}(\phi)$ *is the largest compact ϕ-stable subgroup of G having no proper relatively open ϕ-stable subgroups.*

We know that the family $\mathcal{B}(G)$ of compact open subgroups of G is a local base at e_G and that every $U \in \mathcal{B}(G)$ contains a compact open subgroup that is tidy above for ϕ by Corollary 2.5. Moreover, [19, Corollary 4.3] asserts that a subgroup $U \in \mathcal{B}(G)$ is tidy below for ϕ if and only if $\mathrm{nub}(\phi) \subseteq U$. So we have the following result, where (b) can be deduced from (a) via Lemma 2.1.

Corollary 2.7. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. Then:*

(a) the family of tidy subgroups is a local base at e_G if and only if $\mathrm{nub}(\phi) = \{e_G\}$*;*

(b) if $s(\phi) = 1$, then $\mathrm{nub}(\phi) = \{e_G\}$ *if and only if G has a local base at e_G of ϕ-invariant compact open subgroups.*

According to Theorem 1.1, $\mathrm{nub}(\phi)$ can be obtained also as

$$\mathrm{nub}(\phi) = \bigcap\{U \in \mathcal{B}(G) : U \text{ is minimizing for } \phi\}. \tag{2.3}$$

An advantage of (2.3) is that in case $s(\phi) = 1$, it is much easier to say if a subgroup U is minimizing (i.e., if U is ϕ-invariant), rather than if U is tidy. For example, assume that G is compact; in this case $s(\phi) = 1$, so

$$\mathrm{nub}(\phi) = \bigcap\{U \in \mathcal{B}(G) : U \text{ is } \phi\text{-invariant}\}. \tag{2.4}$$

Now (2.4) allows us to extend the definition of $\mathrm{nub}(\phi)$ also to arbitrary continuous endomorphisms of a compact totally disconnected group G (note that $[\phi(U) : U \cap \phi(U)]$ is finite for every $U \in \mathcal{B}(G)$).

Let us see some examples of computation of the nub.

Example 2.8. (a) Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. If ϕ is periodic (i.e., $\phi^m = id_G$ for some integer $m > 0$), then G has a base of ϕ-invariant compact open subgroups, so $\mathrm{nub}(\phi) = \{e_G\}$ as noted after Corollary 2.7.

(b) If $G = \prod_p N_p$, where p is a prime and each N_p is a finitely generated $\mathbb{J}_p$-module, then G has a base of fully invariant compact open subgroups (namely, $\{mG : m \in \mathbb{N}_+\}$), so $\mathrm{nub}(\phi) = \{e_G\}$ for every continuous endomorphism of G.

(c) Let $G = F^{\mathbb{Z}}$, where F is an arbitrary finite simple group and let $\beta : G \to G$ be the right Bernoulli shift, which is defined by $\beta\big((g_i)_{i\in\mathbb{Z}}\big) = (g_{i-1})_{i\in\mathbb{Z}}$. Then $\mathrm{nub}(\phi) = G$ (see [19, Corollary 4.7]).

(d) Let G be a compact totally disconnected (i.e., profinite) abelian group. Then for every continuous endomorphism $\phi : G \to G$ one can completely describe $\mathrm{nub}(\phi)$ by using the dual endomorphism $\widehat{\phi} : \widehat{G} \to \widehat{G}$ by noting that $\mathrm{nub}(\phi) = t_{\widehat{\phi}}(\widehat{G})^{\perp}$, where $t_{\widehat{\phi}}(\widehat{G})$ is the sum of all finite $\widehat{\phi}$-invariant subgroups of the discrete torsion group $\widehat{G}$ (in terms of [4], $t_{\widehat{\phi}}(\widehat{G})$ is the Pinsker subgroup of $\widehat{\phi}$, i.e., the largest $\widehat{\phi}$-invariant subgroup of $\widehat{G}$ where the restriction of $\widehat{\phi}$ has algebraic entropy zero. According to [4], $\mathrm{nub}(\phi)$ is the greatest ϕ-invariant closed subgroup of G where the restriction of ϕ acts ergodically, i.e., has strongly positive topological entropy (then the induced endomorphism $\overline{\phi} : G/\mathrm{nub}(\phi) \to G/\mathrm{nub}(\phi)$ is the Pinsker factor of ϕ, i.e., $h_{top}(\overline{\phi}) = 0$ and this is the greatest factor with this property, see [4] for more details).

(e) As noted in [19], the dynamical property of the subgroup $\mathrm{nub}(\phi)$ from item (d) remains true in the non-abelian case too. Namely, ϕ acts transitively on $\mathrm{nub}(\phi)$, and $\mathrm{nub}(\phi)$ is the largest closed ϕ-invariant subgroup H_ϕ of G where ϕ acts ergodically.

(f) For any integer $n > 0$ and every topological automorphism $\phi : \mathbb{Q}_p^n \to \mathbb{Q}_p^n$, $\mathrm{nub}(\phi)$ is trivial. Indeed, being a compact subgroup of $\mathbb{Q}_p^n$, $\mathrm{nub}(\phi) \cong \mathbb{J}_p^m$ for some $0 \le m \le n$. By (b) we can conclude that $\mathrm{nub}(\phi)$ has plenty of proper open ϕ-stable subgroups. According to Fact 2.6, this implies $m = 0$.

3 The scale function and the topological entropy

It follows from Corollary 2.4 that

$$\log s(\phi) = \min\{\log[\phi(U_+) : U_+] : U \in \mathcal{B}(G)\}.$$

Furthermore, Theorem 1.2 yields

$$h_{top}(\phi) = \sup\{\log[\phi(U_+) : U_+] : U \in \mathcal{B}(G)\}.$$

This gives the inequality announced in (1.5):

Theorem 3.1. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. Then*

$$\log s(\phi) \le h_{top}(\phi). \tag{3.1}$$

This inequality can be obtained also in another way based on an equivalent definition of the scale function, as explained in the next remark.

Remark 3.2. For G a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism, it was proved in [12, Theorem 7.7] that, for any $U \in \mathcal{B}(G)$,

$$\log s(\phi) = \lim_{n\to\infty} \frac{\log[\phi^n(U) : U \cap \phi^n(U)]}{n}.$$

This gives immediately that $\log s(\phi) \le h_{top}(\phi)$, because $[\phi^n(U) : U \cap \phi^n(U)] = [U : \phi^{-n}(U) \cap U]$ since ϕ is an automorphism, and $[U : \phi^{-n}(U) \cap U] \le [U : U_{-n}]$ as $U_{-n} \subseteq \phi^{-n}(U) \cap U$.

We give now an example witnessing that the inequality in (3.1) can be strict.

Example 3.3. Let p be a prime and $G = \mathbb{Z}(p^\infty)^{\mathbb{Z}}$. Imposing that $U = \mathbb{Z}(p)^{\mathbb{Z}}$ is open and compact in G, then G is given a locally compact (non-compact) topology. Consider $\sigma : G \to G$ the left Bernoulli shift, that is defined by $\sigma\big((g_i)_{i\in\mathbb{Z}}\big) = (g_{i+1})_{i\in\mathbb{Z}}$. Clearly $\sigma(U) = U$, and then:

(a) $s(\sigma) = 1$;

(b) $H_{top}(\sigma, U) = 0$;

(c) $H_{top}(\sigma, V) = \log p$, where $V = \mathbb{Z}(p)^{-\mathbb{N}_+} \oplus \{0\} \oplus \mathbb{Z}(p)^{\mathbb{N}_+}$; in fact $[\sigma(V_+) : V_+] = p$.

Note that $V_+ = \mathbb{Z}(p)^{\mathbb{N}_+}$ and $V_- = \mathbb{Z}(p)^{-\mathbb{N}_+}$, therefore V is tidy above. On the other hand, $V_{++} = \mathbb{Z}(p)^{(-\mathbb{N}_+)} \oplus \{0\} \oplus \mathbb{Z}(p)^{\mathbb{N}_+}$, which is dense in U and so it is not closed; in other words V is not tidy below.

Let G be a totally disconnected locally compact group and $\phi\colon G \to G$ a continuous endomorphism. Since $H_{top}(\phi, -)$ is antimonotone, that is,

$$\text{if } U, V \in \mathcal{B}(G) \text{ and } U \subseteq V, \text{ then } H_{top}(\phi, V) \leq H_{top}(\phi, U),$$

by the definition, it is clear that to compute the topological entropy $h_{top}(\phi)$ it suffices to take the supremum of $H_{top}(\phi, U)$ when U ranges in a local base at e_G of G:

Claim 3.4. *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a continuous endomorphism and $\mathcal{B} \subseteq \mathcal{B}(G)$ a local base at e_G. Then $h_{top}(\phi) = \sup\{H_{top}(\phi, U) : U \in \mathcal{B}\}$.*

Applying this claim on topological entropy, as well as Theorem 1.1 and Theorem 1.2, in the following proposition we give a sufficient condition to have equality in (3.1).

Proposition 3.5. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. If $\mathrm{nub}(\phi) = \{e_G\}$ then $h_{top}(\phi) = \log s(\phi)$.*

Proof. For every tidy $U \in \mathcal{B}(G)$ we have $\log s(\phi) = \log[\phi(U_+) : U_+]$ by Theorem 1.1, so $H_{top}(\phi, U) = \log s(\phi)$ by Theorem 1.2. We are assuming that $\mathrm{nub}(\phi) = \{e_G\}$, so the the tidy subgroups form a local base at e_G by Corollary 2.7. So Claim 3.4 permits to conclude that $h_{top}(\phi) = \log s(\phi)$. □

Example 3.6. For any integer $n > 0$ and every topological automorphism $\phi\colon \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ the equality

$$h_{top}(\phi) = \log s(\phi)$$

holds true. Indeed we know that $\mathrm{nub}(\phi)$ is trivial by Example 2.8(f), so we can conclude using Proposition 3.5.

In other words, Proposition 3.5 says that, if the inequality in (3.1) is strict, then $\mathrm{nub}(\phi) \neq \{e_G\}$. This is the case of topological automorphisms of compact groups with positive topological entropy. Actually, one can say more in the case of compact groups, since we know that $s(\phi) = 1$ if and only if there exists a ϕ-invariant $U \in \mathcal{B}(G)$ by Lemma 2.1(d).

Let us work now under the blanket assumption that $s(\phi) = 1$ (i.e., there exists a ϕ-invariant $U \in \mathcal{B}(G)$ by Lemma 2.1(d)). From (3.1) we deduce that $h_{top}(\phi) = \log s(\phi)$ precisely when $h_{top}(\phi) = 0$. Moreover, we have the following

Theorem 3.7. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism with $s(\phi) = 1$. Then the following conditions are equivalent:*

(a) $h_{top}(\phi) = \log s(\phi)$*;*

(b) $H_{top}(\phi, U) = 0$ *for all* $U \in \mathcal{B}(G)$*;*

(c) G has a local base at e_G formed by ϕ-invariant $U \in \mathcal{B}(G)$;

(d) G has a local base at e_G formed by subgroups tidy for ϕ;

(e) $\mathrm{nub}(\phi) = \{e_G\}$.

Proof. The implication (e)⇒(a) holds by Proposition 3.5, while (a)⇒(b) and (c)⇒(d)⇒(e) are obvious. To see that (b)⇒(c) note that $H_{top}(\phi, U) = 0$ for some $U \in \mathcal{B}(G)$ implies that there exists an integer $n \geq 0$ such that $U_{-n} = U_-$; then $U_- \in \mathcal{B}(G)$, $U_- \subseteq U$ and it is ϕ-invariant. □

The conditions of Theorem 3.7 are satisfied in obvious way when the group G is either compact or discrete. On the other hand, Example 3.3 furnishes the non-trivial example when $s(\phi) = 1$ but ϕ does not satisfy the condition in item (a).

4 Basic "entropic" properties of the scale function

In this section we give properties of the scale function similar to the basic properties satisfied by the topological entropy; so we start reminding the latter ones in the following result.

Fact 4.1. *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism.*

(a) [Invariance under conjugation] *If H is another totally disconnected locally compact group and $\xi : G \to H$ is a topological isomorphism, then $h_{top}(\phi) = h_{top}(\xi\phi\xi^{-1})$.*

(b) [Logarithmic Law] *For every integer $k \geq 0$ we have $h_{top}(\phi^k) = k \cdot h_{top}(\phi)$.*

(c) [Monotonicity] *If H is a closed ϕ-stable subgroup of G, then $h_{top}(\phi) \geq h_{top}(\phi \restriction_H)$; if H is normal and $\overline{\phi} : G/H \to G/H$ is the topological automorphism induced by ϕ, then $h_{top}(\overline{\phi})$.*

(d) [Weak Addition Theorem] *If $G = G_1 \times G_2$ and $\phi_i : G_i \to G_i$ is a topological automorphism for $i = 1, 2$, then $h_{top}(\phi_1 \times \phi_2) = h_{top}(\phi_1) + h_{top}(\phi_2)$.*

(e) [Continuity] *If G is an inverse limit $G = \varprojlim G/N_i$ with N_i closed normal ϕ-stable subgroup, then $h_{top}(\phi) = \sup_{i\in I} h_{top}(\overline{\phi}_i)$, where $\overline{\phi}_i : G/N_i \to G/N_i$ is the topological automorphism induced by ϕ.*

Invariance under conjugation is clear also for the scale function:

Lemma 4.2 (Invariance under conjugation)**.** *Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. Let H be another totally disconnected locally compact group and $\xi : G \to H$ a topological isomorphism. Then $s(\phi) = s(\xi\phi\xi^{-1})$.*

Proof. It is easy to see that $s_G(\phi, U) = s_H\big(\xi\phi\xi^{-1}, \xi(U)\big)$ for every $U \in \mathcal{B}(G)$ and that $\mathcal{B}(H) = \{\xi(U) : U \in \mathcal{B}(G)\}$. □

Fact 4.3 (Logarithmic law)**.** [15, Corollary 3] *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and $n \geq 0$ an integer. Then $s(\phi^n) = s(\phi)^n$.*

Monotonicity was proved by Willis, indeed he proves the following more precise relation.

Fact 4.4 (Monotonicity)**.** [17, Proposition 4.7] *Let G be a totally disconnected locally compact group, $\phi : G \to G$ a topological automorphism and H a closed subgroup of G such that $\phi(H) = H$, then*

(a) *$s(\phi) \geq s(\phi \restriction_H)$.*

If H is also normal let $\overline{\phi} : G/H \to G/H$ be the topological automorphism induced by ϕ, then

(b) *$s(\phi \restriction_H) \cdot s(\overline{\phi})$ divides $s(\phi)$.*

Remark 4.5. (a) We call the property in item (d) of Fact 4.1 Weak Addition Theorem. Indeed, the stronger so-called Addition Theorem holds for the topological entropy in the compact case (see [3, 20]); more precisely, if G is a compact group, $\phi : G \to G$ is a continuous endomorphism and N is a closed ϕ-invariant normal subgroup of G, then

$$h_{top}(\phi) = h_{top}(\phi \restriction_N) + h_{top}(\overline{\phi}),$$

where $\overline{\phi} : G/N \to G/N$ is the continuous endomorphism induced by ϕ.

(b) It is not known whether the Addition Theorem for the topological entropy holds also in the general case of locally compact groups, even under the hypotheses that G is totally disconnected (and abelian) and that $\phi : G \to G$ is a topological automorphism.

(c) The counterpart of the Addition Theorem for the scale function does not hold true in general, since [17, Example 6.4] shows that the inequality $s(\phi) \geq s(\phi \restriction_H) \cdot s(\overline{\phi})$ can be strict.

On the other hand, we see in Theorem 4.6 below that the Weak Addition Theorem holds also for the scale function.

Note that we call this kind of properties Addition Theorem also for the scale function, even if they have a multiplicative form in this case; just take the logarithm to have the additive form.

Theorem 4.6 (Weak Addition Theorem)**.** *Let G, H be locally compact totally disconnected groups and $\phi : G \to G$, $\psi : H \to H$ topological automorphisms. Then $s(\phi \times \psi) = s(\phi) \cdot s(\psi)$.*

Proof. Let $V \in \mathcal{B}(G)$ be tidy for ϕ and $W \in \mathcal{B}(H)$ be tidy for ψ. For the compact and open subgroup $V \times W \subseteq G \times H$, we have that

$$\begin{aligned}(V \times W)_+ &= \bigcap_{k\geq 0} (\phi \times \psi)^k (V \times W) = \bigcap_{k\geq 0} \big(\phi^k(V) \times \psi^k(W)\big) \\ &= \Big(\bigcap_{k\geq 0} \phi^k(V)\Big) \times \Big(\bigcap_{k\geq 0} \psi^k(W)\Big) = V_+ \times W_+,\end{aligned}$$

and in the same way one can prove that $(V \times W)_- = V_- \times W_-$. Since

$$(V \times W)_+(V \times W)_- = (V_+V_-) \times (W_+W_-) = V \times W,$$

we have that $V \times W$ is tidy above for $\phi \times \psi$. The subgroup $V \times W$ is also tidy below for $\phi \times \psi$ because

$$\begin{aligned}(V \times W)_{++} &= \bigcup_{k\geq 0} (\phi \times \psi)^k (V \times W)_+ = \bigcup_{k\geq 0} \Big(\phi^k(V_+) \times \psi^k(W_+)\Big) \\ &\overset{*}{=} \Big(\bigcup_{k\geq 0} \phi^k(V_+)\Big) \times \Big(\bigcup_{k\geq 0} \psi^k(W_+)\Big) = V_{++} \times W_{++}\end{aligned}$$

is a closed subgroup of $G \times H$. The equality $(*)$ holds because the families $\{\phi^k(V_+)\}_{k\geq 0}$ and $\{\psi^k(W_+)\}_{k\geq 0}$ are increasing families of subgroups of G and H respectively.

Therefore $V \times W$ is tidy for $\phi \times \psi$ and

$$s(\phi \times \psi) = \big[(\phi \times \psi)(V \times W) : \big((V \times W) \cap (\phi \times \psi)(V \times W)\big)\big].$$

This index is equal to

$$\big[\phi(V) : V \cap \phi(V)\big] \cdot \big[\psi(W) : W \cap \psi(W)\big] = s(\phi) \cdot s(\psi)$$

and hence $s(\phi \times \psi) = s(\phi) \cdot s(\psi)$. □

Proposition 4.7 (Continuity for direct limits)**.** *Let $\phi\colon G \to G$ be a topological automorphism and $G \cong \varinjlim_{i\in I} H_i$ a totally disconnected locally compact group, where $\{H_i\}_{i\in I}$ is a directed system of open ϕ-stable subgroups of G. Then exists $j \in I$ such that*

$$s(\phi) = s(\phi \restriction_{H_j}) = \max_{i\in I} s(\phi \restriction_{H_i}).$$

Proof. By Lemma 4.4(a) the inequalities

$$s(\phi) \geq s(\phi \restriction_{H_i}) \tag{4.1}$$

hold true for every $i \in I$.

Let $U \in \mathcal{B}(G)$ be a minimizing subgroup for ϕ. Then $\{H_i \cap U\}_{i\in I}$ is an open covering of U and so admits a finite open subcover, because U is compact. This means that exists a finite set $F \subseteq I$ such that $U \subseteq \bigcup_{i\in F} H_i$. Moreover, there exists an index $j \in I$ such that $H_i \subseteq H_j$ for every $i \in F$ and so $U \subseteq H_j$. In particular, $U \in \mathcal{B}(H_j)$. This implies that U is tidy above also for $\phi \restriction_{H_j}$. Indeed, both automorphisms ϕ and $\phi \restriction_{H_j}$ share the same subgroups U_+ and U_-. Moreover, U is tidy below for $\phi \restriction_{H_j}$, because

$$U_{\phi\restriction_{H_j}++} = U_{\phi++} = U_{\phi++} \cap H_j$$

is a closed subgroup of H_j. This means that

$$s(\phi) = s(\phi, U) = s(\phi \restriction_{H_j}, U) = s(\phi \restriction_{H_j}).$$

This means, in view of (4.1), that

$$s(\phi) = s(\phi \restriction_{H_j}) = \max_{i\in I} s(\phi \restriction_{H_i}).$$

□

Remark 4.8. (a) A counterpart of Proposition 4.7 regarding continuity for inverse limits holds true: if a totally disconnected locally compact group $G = \varprojlim G/N_i$ is an inverse limit and $\phi : G \to G$ is a topological automorphism such that the closed normal ϕ-stable subgroups N_i for all $i \in I$, then $s(\phi) = \max_{i\in I} s(\overline{\phi}_i)$, where $\overline{\phi}_i : G/N_i \to G/N_i$ is the topological automorphism induced by ϕ. A proof can be found in [2], the case of inner automorphisms is proved in [17, Proposition 5.4]. As shown in [2], the general case can be deduced also from the case of inner automorphisms by means of the natural property given in item (b).

(b) ([2]) Let G be a totally disconnected locally compact group and $\phi : G \to G$ a topological automorphism. Then $s_G(\phi) = s_{G_1}(\phi_\ltimes)$, where $\phi_\ltimes$ is the inner automorphism of $G_1 = \langle\phi\rangle \ltimes G$ defined by the element (ϕ, e_G).

Applying Proposition 4.7 one obtains the following corollary still concerning continuity of the scale function with respect to direct limits; the condition on the stable subgroups is relaxed from open to closed, while the set of indices is now supposed to be countable.

Corollary 4.9. *Let G be a totally disconnected locally compact group and $\phi\colon G \to G$ a topological automorphism. If $G \cong \varinjlim_{n\geq 0} H_n$, where $\{H_n\}_{n\geq 0}$ is a directed system of closed ϕ-stable subgroups of G, then exists an integer $n \geq 0$ such that*

$$s(\phi) = s(\phi\restriction_{H_n}) = \max_{n\in\mathbb{N}} s(\phi\restriction_{H_n}).$$

Proof. Apply the Baire Category Theorem to $G = \bigcup_{n\in\mathbb{N}} H_n$ to conclude that for some $m \geq 0$ the subgroup H_m has non-empty interior. Therefore, H_n is open for all $n \geq m$. Now apply Proposition 4.7 to the family of these open subgroups. □

5 Scalar p-adic Yuzvinski Formula and Bridge Theorem

In the first part of this section we calculate directly the value of the scale function for any topological automorphism $\phi : \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ in Theorem 5.3; note that ϕ is a $\mathbb{Q}_p$-linear transformation.

We start recalling some useful information about the p-adic numbers, giving reference to [9] for more details. Let $|-|_p$ be the p-adic norm over $\mathbb{Q}_p$, i.e., for $\xi \in \mathbb{Q}_p$

$$|\xi|_p = \begin{cases} 0 & \text{if } \xi = 0, \\ p^r & \text{if } \xi = p^{-r}\left(\sum_{i=0}^{\infty} a_i p^i\right) \text{ with } a_0 \in \{1, 2, \dots, p-1\}. \end{cases}$$

With $\mathbb{J}_p$ we indicate the p-adic integers, that is, $\mathbb{J}_p = \{\xi \in \mathbb{Q}_p : |\xi|_p \leq 1\}$. This is a local PID with maximal ideal $\{\xi \in \mathbb{Q}_p : |\xi|_p < 1\}$.

If K is a finite extension of $\mathbb{Q}_p$ of degree $d = [K : \mathbb{Q}_p]$, then the p-adic norm $|-|_p$ over $\mathbb{Q}_p$ can be extended to a norm over K, and this extension is unique. We indicate this extended norm with $|-|_p$ and we call it the p-adic norm over K. Let

$$\mathcal{O} = \{\xi \in K : |\xi|_p \leq 1\};$$

then $\mathcal{O}$ is a local PID with maximal ideal

$$\mathfrak{m} = \{\xi \in K : |\xi|_p < 1\}.$$

Consider now a generator π' for $\mathfrak{m}$, then there exists an integer $e > 0$ such that $p = u\pi'^e$, where u is an unit in $\mathbb{J}_p$. Denote with π a generator of $\mathfrak{m}$ such that $p = \pi^e$. This number e is independent of the choice of the generator of $\mathfrak{m}$, divides the degree d of the extension and it is called the *ramification index* of K over $\mathbb{Q}_p$. One can prove that the residual field $\mathcal{O}/\mathfrak{m}$ has cardinality

$$\left|\mathcal{O}/\mathfrak{m}\right| = [\mathcal{O} : \mathfrak{m}] = p^f, \tag{5.1}$$

where we let $f = d/e$.

Example 5.1. Let K be a finite extension of $\mathbb{Q}_p$ of degree d, $\lambda \in K$ and consider the topological automorphism $\phi\colon K^n \to K^n$ defined by the Jordan block

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda & 1 \\ 0 & \dots & 0 & 0 & \lambda \end{pmatrix} \in GL_n(K).$$

We see that $\mathcal{O}^n$ is minimizing for ϕ and that

$$s_{K^n}(\phi) = \max\{1, {|\lambda|_p}^{dn}\}.$$

Note that the nature of the automorphism ϕ is completely determined by λ. In fact if $\lambda \in \mathcal{O}$ then $\mathcal{O}^n$ is ϕ-invariant and $s(\phi) = 1$, otherwise it is inversely ϕ-invariant. In both cases the subgroup is minimizing for ϕ by Lemma 2.1(a,c).

Suppose that $\lambda \notin \mathcal{O}$, this means that

$$s(\phi) = \big[\phi(\mathcal{O}^n) : \mathcal{O}^n \cap \phi(\mathcal{O}^n)\big] = \big[(\lambda\mathcal{O})^n : \mathcal{O}^n\big] = \big[\lambda\mathcal{O} : \mathcal{O}\big]^n.$$

Let e the ramification index of the extension K over $\mathbb{Q}_p$, π a generator for $\mathfrak{m}$ such that $p = \pi^e$ and $f = d/e$. Then $\lambda = \pi^{-l}\xi$, where $\xi \in \mathcal{O} \setminus \mathfrak{m}$, $l > 0$ and

$$|\lambda|_p = |\pi^{-l}|_p = |p^{-l/e}|_p = p^{l/e}.$$

This yields

$$s(\phi) = \big[\lambda\mathcal{O} : \mathcal{O}\big]^n = \big[\pi^{-l}\mathcal{O} : \mathcal{O}\big]^n = [\pi^{-1}\mathcal{O} : \mathcal{O}]^{ln}.$$

By (5.1), $[\pi^{-1}\mathcal{O} : \mathcal{O}] = p^f$ and so

$$s(\phi) = (p^f)^{ln} = (p^{l/e})^{efn} = {|\lambda|_p}^{dn}.$$

In conclusion

$$s_{K^n}(\phi) = \max\{1, {|\lambda|_p}^{dn}\}.$$

This example allows us to obtain an explicit formula for the scale function on $\mathbb{Q}_p^n$, as we shall see in Theorem 5.3.

Example 5.2. Let $\phi\colon \mathbb{Q}_p^2 \to \mathbb{Q}_p^2$ the topological automorphism defined by the matrix $\begin{pmatrix} 0 & p \\ p^{-1} & 0 \end{pmatrix}$. Then $\phi^2 = id$ and so $s(\phi) = 1$ by Fact 4.3.

Nevertheless

$$s(\phi, \mathbb{J}_p^2) = \big[(p\mathbb{J}_p) \times (p^{-1}\mathbb{J}_p) : (p\mathbb{J}_p) \times (\mathbb{J}_p)\big] = p$$

and hence $\mathbb{J}_p^2$ is not a minimizing subgroup for ϕ, although it is a minimizing subgroup for the canonical Jordan form of ϕ. Indeed

$$\begin{pmatrix} 0 & p \\ p^{-1} & 0 \end{pmatrix} = \begin{pmatrix} p & -p \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} p^{-1}/2 & 1/2 \\ -p^{-1}/2 & 1/2 \end{pmatrix}$$

and if we denote by $\psi\colon \mathbb{Q}_p^2 \to \mathbb{Q}_p^2$ the topological automorphism defined by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, it is easy to see that $s(\psi, \mathbb{J}_p^2) = 1$.

This means that, though ψ is coniugated to ϕ by ξ (defined by the matrix $\begin{pmatrix} p & -p \\ 1 & 1 \end{pmatrix}$), there exist subgroups minimizing for the first automorphism that are not minimizing for the second.

Note that Example 5.2 is not in contradiction with Lemma 4.2. Indeed, the correspondence between minimizing subgroups of ϕ and minimizing subgroups of ψ is given by the map $U \mapsto \xi(U)$, while $\phi = \xi\psi\xi^{-1}$.

The next formula (5.2) was pointed out (without proof) in the Introduction of [18] in the case when all eigenvalues belong to $\mathbb{Q}_p$. In the general case, by Example 3.6 we have $h_{top}(\phi) = \log s(\phi)$, so one could apply the p-adic Yuzvinski Formula for the topological entropy proved in [11] to obtain (5.2). Nevertheless, we give a direct proof of this formula for sake of completeness, but also because the computation of the scale function is simpler than that of the topological entropy; indeed, for the scale function it suffices to take into account only one compact open subgroup which is minimizing (i.e., tidy) for ϕ, without any recourse to the Haar measure.

Theorem 5.3. *Let $\phi\colon \mathbb{Q}_p^n \to \mathbb{Q}_p^n$ be a topological automorphism, for an integer $n > 0$. Then*

$$s_{\mathbb{Q}_p^n}(\phi) = \prod_{|\lambda_i|_p > 1} |\lambda_i|_p, \tag{5.2}$$

where $\{\lambda_1, \ldots, \lambda_n\}$ is the family of all eigenvalues of ϕ contained in a finite extension K of $\mathbb{Q}_p$.

Proof. Let $\lambda_1, \ldots, \lambda_r$ be all the distinct eigenvalues of ϕ and assume without loss of generality that $K = \mathbb{Q}_p[\lambda_1, \ldots, \lambda_r]$, that is, K is the splitting field of the minimal polynomial of ϕ over $\mathbb{Q}_p$.

Let $\phi^K = \phi \otimes_{\mathbb{Q}_p} id_K\colon K^n \to K^n$, where $\otimes_{\mathbb{Q}_p}$ is the tensor product over $\mathbb{Q}_p$. The automorphisms ϕ and ϕ^K are represented by the same matrix respectively over $\mathbb{Q}_p$ and K, hence they have the same eigenvalues.

Let $\mathcal{A}$ be a base of K over $\mathbb{Q}_p$, then every $\xi \in K$ has coordinates $[\xi]_{\mathcal{A}} = (\xi_{(1)}, \ldots, \xi_{(d)})$. Moreover $K^n \cong \mathbb{Q}_p^{dn}$ and this isomorphism $\alpha\colon K^n \to \mathbb{Q}_p^{dn}$ is given by

$$\alpha(\xi_1, \ldots, \xi_n) = \Big([\xi_1]_{\mathcal{A}}, \ldots, [\xi_n]_{\mathcal{A}}\Big).$$

Let

$$\Phi = \underbrace{\phi \times \cdots \times \phi}_{d \text{ times}}\colon \mathbb{Q}_p^{dn} \to \mathbb{Q}_p^{dn};$$

then $\phi^K = \alpha^{-1}\Phi\alpha$. Lemma 4.2 and Fact 4.3 yield

$$s_{\mathbb{Q}_p^n}(\phi)^d = s_{\mathbb{Q}_p^{dn}}(\Phi) = s_{K^n}(\phi^K). \tag{5.3}$$

In K^n the automorphism ϕ^K has a canonical Jordan form. Let $J_i \in GL_{m_i}(K)$ be the Jordan block associated to the eigenvalue λ_i, where n_i is the multiplicity of λ_i. Define $\phi_i\colon K^{n_i} \to K^{n_i}$ as the topological automorphism associated to J_i, for every $i = 1, \ldots, r$. Then, by Theorem 4.6,

$$s_{K^n}(\phi^K) = s_{K^{n_1}}(\phi_1) \cdot \ldots \cdot s_{K^{n_r}}(\phi_r). \tag{5.4}$$

From (5.3) and (5.4), and from Example 5.1, we obtain that

$$s_{\mathbb{Q}_p^n}(\phi)^d = \prod_{|\lambda_i|_p > 1} {|\lambda_i|_p}^{dn_i}.$$

This gives immediately the thesis. $\square$

In the second part of this section we provide a so-called Bridge Theorem for the scale function. To this end we first recall some properties of the Pontryagin duality.

We say that a locally compact group G is *compactly covered*, if every element of G is contained in some compact subgroup of G. The following folklore fact can be easily deduced from the standard properties of locally compact groups.

Fact 5.4. *For a locally compact abelian group G, the following are equivalent:*

(a) G is compactly covered;

(b) contains no copies of the discrete group $\mathbb{Z}$;

(c) there exist no continuous surjective homomorphisms $\widehat{G} \to \mathbb{T}$;

(d) $\widehat{G}$ is a totally disconnected.

Our interest in Fact 5.4 stems from the necessity to describe the class of totally disconnected locally compact abelian groups G, such that dual group $\widehat{G}$ is totally disconnected as well. According to the above fact, these are precisely the compactly covered totally disconnected locally compact abelian groups. Therefore, for such a group G one can define the scale function on the dual group $\widehat{G}$, and we are interested in the relationship between $s_G(\phi)$ and $s_{\widehat{G}}(\widehat{\phi})$, where $\phi\colon G \to G$ is a topological automorphism of G.

We collect now two known facts that apply in the proof of Theorem 5.7.

Fact 5.5. *Let G be a locally compact abelian group.*

(a) If G is finite then $G \cong \widehat{G}$.

(b) If G is discrete then $\widehat{G}$ is compact, and vice versa.

Recall that, if $X \subseteq G$, the annihilator of X in $\widehat{G}$ is

$$X^{\perp} = \{\chi \in \widehat{G} : \chi(x) = 0\, \forall x \in X\}$$

and, if $Y \subseteq \widehat{G}$, the annihilator of Y in G is

$$Y^{\perp} = \{g \in G : \chi(g) = 0\, \forall \chi \in Y\}.$$

Fact 5.6. *Let G be a locally compact abelian group, U a compact subgroup of G and $\phi\colon G \to G$ an automorphism. Then:*

(a) $(U^{\perp})^{\perp} = U$;

(b) $U^{\perp} \cong \widehat{G/U}$ and $\widehat{U} \cong \widehat{G}/U^{\perp}$;

(c) $\big(U + \phi(U)\big)^{\perp} = U^{\perp} \cap \widehat{\phi}^{-1}(U^{\perp})$;

(d) if V is another compact subgroup of G and $U \subseteq V$, then $\widehat{V/U} \cong U^{\perp}/V^{\perp}$.

We are now in the conditions to prove the next theorem, which asserts that $s_{\widehat{G}}(\widehat{\phi})$ is equal to $s_G(\phi)$.

Theorem 5.7. *Let $\phi\colon G \to G$ be a topological automorphism of a totally disconnected compactly covered locally compact abelian group G. Then $s_{\widehat{G}}(\widehat{\phi}) = s_G(\phi)$.*

Proof. According to Fact 5.4, the group $\widehat{G}$ is totally disconnected, so one can define the scale function on $\widehat{G}$.

We first check that $U \in \mathcal{B}(G)$ if and only if $U^{\perp} \in \mathcal{B}(\widehat{G})$. In fact, assume that $U \in \mathcal{B}(G)$. Then G/U is discrete because U is open in G, and so $\widehat{G/U}$ is a compact group. Since $U^{\perp} \cong \widehat{G/U}$ by Fact 5.6, so $U^{\perp}$ is a compact subgroup of $\widehat{G}$. Moreover, U is compact in G and so, since $\widehat{G}/U^{\perp} \cong \widehat{U}$ by Fact 5.6, $\widehat{G}/U^{\perp}$ is discrete; therefore, $U^{\perp}$ is open in $\widehat{G}$. Hence, we have proved that $U \in \mathcal{B}(G)$ implies $U^{\perp} \in \mathcal{B}(\widehat{G})$.

To verify the converse implication it suffices to note that $G \cong \widehat{\widehat{G}}$ canonically by Pontryagin duality and that $(U^{\perp})^{\perp} = U$, and then apply the previous implication. With the same argument one can prove that if $V \in \mathcal{B}(\widehat{G})$, then $V^{\perp} \in \mathcal{B}(G)$, as $(V^{\perp})^{\perp} = V$ by Fact 5.6,

Let us see that the equality $s(\phi, U) = s(\widehat{\phi}, U^{\perp})$ holds true. In fact, by the Second Isomorphism Theorem, $U/\big(U \cap \phi(U)\big) \cong \big(U + \phi(U)\big)/U$ and so

$$s(\phi, U) = \Big|\frac{U}{U \cap \phi(U)}\Big| = \Big|\frac{U + \phi(U)}{U}\Big|. \tag{5.5}$$

Moreover, let $F = \big(U + \phi(U)\big)/U$; then F is a finite group and $F \cong \widehat{F}$ by Fact 5.5. Therefore, by (5.5) and Fact 5.6, we have that

$$s(\phi, U) = |F| = |\widehat{F}| = \Big|\frac{U^{\perp}}{(U + \phi(U))^{\perp}}\Big| = \Big|\frac{U^{\perp}}{U^{\perp} \cap \widehat{\phi}^{-1}(U^{\perp})}\Big|. \tag{5.6}$$

Finally

$$\Big|\frac{U^{\perp}}{U^{\perp} \cap \widehat{\phi}^{-1}(U^{\perp})}\Big| = \Big|\frac{\widehat{\phi}(U^{\perp})}{\widehat{\phi}(U^{\perp}) \cap U^{\perp}}\Big| = s(\widehat{\phi}, U^{\perp}), \tag{5.7}$$

since $\widehat{\phi}$ is an automorphism. Equations (5.6) and (5.7) imply the equality

$$s(\phi, U) = s(\widehat{\phi}, U^{\perp}). \tag{5.8}$$

Now let U be a minimizing subgroup for ϕ. If $V \in \mathcal{B}(\widehat{G})$, then $V^\perp \in \mathcal{B}(G)$ by the first part of the proof, and $s(\phi, V^\perp) = s(\widehat{\phi}, V)$ by (5.8). So, applying twice (5.8), we have

$$s(\widehat{\phi}, U^\perp) = s(\phi, U) \leq s(\phi, V^\perp) = s(\widehat{\phi}, V).$$

Since this inequality holds true for all $V \in \mathcal{B}(\widehat{G})$, we can conclude that $U^\perp$ is a minimizing subgroup for $\widehat{\phi}$, hence

$$s(\widehat{\phi}) = s(\widehat{\phi}, U^\perp) = s(\phi, U) = s(\phi)$$

by (5.8) and the assumption that U is minimizing for ϕ. $\square$

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