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Planar loops with prescribed curvature: existence, multiplicity and uniqueness results

Roberta Musina*

Abstract. Let $k : \mathbb{C} \to \mathbb{R}$ be a smooth given function. A k-loop is a closed curve u in $\mathbb{C}$ having prescribed curvature $k(p)$ at every point $p \in u$. We use variational methods to provide sufficient conditions for the existence of k-loops. Then we show that a breaking symmetry phenomenon may produce multiple k-loops, in particular when k is radially symmetric and somewhere increasing. If $k > 0$ is radially symmetric and non increasing we prove that any embedded k-loop is a circle, that is, round circles are the only convex loops in $\mathbb{C}$ whose curvature is a non increasing function of the Euclidean distance from a fixed point. The result is sharp, as there exist radially increasing curvatures $k > 0$ which have embedded k-loops that are not circles.

Keywords: Plane curves, prescribed curvature.

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Introduction

In this paper we prove existence, multiplicity and uniqueness results for the following k-loop problem: *given a non constant function $k : \mathbb{C} \to \mathbb{R}$ find k-loop, that is, a closed curve u in $\mathbb{C}$ having prescribed curvature $k(p)$ at every point $p \in u$.*

Several papers deal with loops with prescribed curvature and with related questions, see for example [5]-[7], [12]-[14], [18], [19]. For the correspondent problem in dimension 3, the H-bubble problem for closed surfaces in $\mathbb{R}^3$ with prescribed mean curvature, we quote [4], [8]-[11], [19], [20] and references there-in.

In Section 1 we provide global sufficient conditions for the existence of k-loops by studying

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the ordinary differential system

$$\begin{cases} u'' = \left(\int_0^1 |u'|^2\right)^{1/2} k(u)(iu') \quad \text{on} \quad (0,1) \\ u(0) = u(1)\,. \end{cases} \tag{0.1}$$

Any non constant weak solution u to (0.1) is a smooth periodic function on $\mathbb{R}$ by regularity theory. As u'' is orthogonal to u', then $|u'|$ is a constant. Thus $u'' \cdot (iu') = |u'|^3 k(u)$, that is, u is a k-loop.

If the prescribed curvature $k \in C^2(\mathbb{C})$ satisfies

(k_1) $M_k := \sup_{z\in\mathbb{C}} |(\nabla k(z) \cdot z)\; z| < 1$,

(k_2) there exists a constant $k_\infty > 0$ such that $k(z) = k_\infty + o(|z|^{-1})$ as $|z| \to \infty$,

then problem (0.1) can be plainly studied via variational methods. Indeed, solutions to (0.1) are critical points of an energy functional $\mathcal{E}_k$ defined on the space H_{per} of 1-periodic functions in $H^1(0,1)$ and assumptions (k_1) and (k_2) guarantees that $\mathcal{E}_k$ enjoys good regularity and geometrical properties (see Section 1 for details). In particular the Nehari manifold

$$\Sigma_k = \left\{ u \in H_{per} \mid u \text{ is non constant },\ \mathcal{E}_k'(u) \cdot u = 0 \right\}$$

is a not empty and smooth submanifold of H_{per}. In addition, the functional $\mathcal{E}_k$ is positively bounded from below on Σ_k by (k_1) and by the isoperimetric inequality.

In despite of the non homogeneous nature of problem (0.1), it turns out that Σ_k is a natural constraint for $\mathcal{E}_k$. This means that if u is a critical point for $\mathcal{E}_k$ on Σ_k then u solves (0.1), and hence u is a k-loop. Thus we are lead to study the minimization problem

$$\underline{c} := \inf_{u\in\Sigma_k} \mathcal{E}_k(u)\,. \tag{0.2}$$

Any solution to (0.1) belongs to Σ_k. A solution u to (0.1) is said to be a *minimal k-loop* if it solves (0.2), that is, if it has the minimal energy among all k-loops.

The infimum $\underline{c}$ might be not achieved, because of the lack of compactness produced by the group of translations $u \mapsto u + p$. On the other hand, it turns out that $c \le \pi/k_\infty$ by Lemma 1.4 and that the Palais Smale condition for $\mathcal{E}_k$ on Σ_k holds at any energy level c which is not a multiple of π/k_∞ (see Lemma 1.7). Therefore, a minimal k-loop exists provided that $\underline{c} < \pi/k_\infty$ (compare with Theorems 1.9 and 1.11).

If the minimal energy $\underline{c}$ is not achieved in Σ_k, then Lemma 1.7 and a careful analysis of the topological properties of the energy sublevels allow us to find sufficient conditions for the existence of an higher unstable critical point (see Subsection 1.2). The argument was inspired by the paper [11] on the H-bubble problem.

The main results in the present paper highlight the connection between the monotonicity properties of the prescribed curvature k along radial directions and the number of geometrically distinct k-loops.

Multiple k-loops may be produced by a breaking symmetry phenomenon for problem (0.2). Assume that k satisfies the assumptions (k_1), (k_2) and that $Rk(z) \equiv 1$ for some $R > 0$ and for any $z \in C_R := \{|z| = R\}$. This always happens, for instance, when k is radially symmetric about the origin. We parametrize the circle C_R by the map $t \mapsto Re^{2\pi it}$, so that C_R can be regarded as a k-loop in H_{per}. We wonder if C_R solves (0.2). In Section 2 we show that the monotonicity properties of k on C_R along the normal directions to C_R affect the minimality of the circle C_R. In case of a radially symmetric curvature we obtain the following result (see Theorem 2.1 for a more general statement).

Theorem 0.1 *Assume that $k \in C^2(\mathbb{C})$ is a radially symmetric curvature satisfying* (k_1) *and* (k_2)*. If at the radius $R > 0$ it holds that $Rk(R) = 1$ and $k'(R) > 0$, then the circle C_R is not a minimal k-loop.*

Motivated by Theorem 0.1, in the last part of the paper we restrict our attention to radially symmetric curvatures. It is straightforward to notice that if k depends only on the distance from the origin and if u is a k-loop then $\mathcal{R} \circ u$ is a k-loop for any rotation $\mathcal{R}$ of the complex plane. Thus, any not round k-loop generates a 1-dimensional manifold of distinct k-loops.

As a consequence of Theorem 0.1, in Section 3 we obtain a sufficient condition for the existence of a round k-loop and of a family of minimal k-loops which are not circles (compare with Corollary 3.1).

Breaking symmetry may occur also in a more general setting. In Example 3.2 we exhibit a radially increasing curvature $k > 0$ such that $M_k > 1$, and such that there exists an embedded k-loop which is not a circle.

To better understand the breaking symmetry and the multiplicity phenomena in the k-loop problem, in Proposition 3.3 we prove that the existence of a round k-loop is a necessary condition for the existence of k-loops. As a consequence of Proposition 3.3, in Corollary 3.4 we point out a nonexistence result.

In Section 4 we prove the following theorem, that underlines once again the connection between the multiplicity phenomenon and the monotonicity properties of k.

Theorem 0.2 *Let $k \in C^0(\mathbb{C})$ be a positive and radially symmetric function. Assume that k is non-increasing as a function of the distance from the origin. Then any embedded k-loop is a circle.*

The result is sharp, in view of Corollary 3.1 and Example 3.2.

The main tools in the proof are a *Touching Lemma*, that follows from Hopf's maximum principle, and Osserman's construction for the four vertex theorem.

Several uniqueness (up to homothety) results are available for similar geometrical problems. A complete list of references would lead us far from our purposes. We quote the pioneering papers [2] by Alexandrov and [1] by Aeppli, where the prescribed curvature is assumed to be homogeneous of degree -1. Treibergs and Wei [19] proved the uniqueness of embedded radial graphs over the unit sphere $\mathbb{S}^n$ having positive prescribed mean curvature $H \in C^1(\mathbb{R}^{n+1})$, such that $H(p) + \nabla H(p) \cdot p \leq 0$ for any $p \in \mathbb{R}^{n+1}$.
A result like Theorem 0.2 is missing for the H-bubble problem. It would be of interest to know if spheres are the only convex surfaces in $\mathbb{R}^3$ whose mean curvature H is a non increasing function of the distance from the origin.

Notation. Let z_1, z_2 be two points in the complex plane $\mathbb{C}$. We denote by $z_1 z_2 \in \mathbb{C}$ and by $z_1 \cdot z_2 \in \mathbb{R}$ their complex and scalar products, respectively. Let $R > 0$ and $z \in \mathbb{C}$. The open disk of center z and radius R is denoted by $D_R(z)$. If $z = 0$ we simply write D_R instead of $D_R(0)$. The unit circle $\mathbb{S}^1 = \partial D_1$ is parametrized by the function

$$\omega(t) := e^{2\pi i t} \ , \quad \omega : (0,1) \to \mathbb{C} \, . \tag{0.3}$$

A *curve* or parametrized arc in $\mathbb{C}$ is a complex-valued map g of class C^2 defined on an open interval I of $\mathbb{R}$, such that $g'(t) \neq 0$ for any $t \in I$. We will often identify the curve g with its image. The (signed) *curvature* of g at the point $g(t)$ is given by

$$K_{g(t)} = \frac{g''(t) \cdot (i g'(t))}{|g'(t)|^3} \ . \tag{0.4}$$

Every curve admits a parametrization $\bar{g}$ by arclength (see for example [3], Proposition 8.3.2). Since $|\bar{g}'| \equiv 1$ then $K_{\bar{g}} \equiv \bar{g}'' \cdot (i\bar{g}')$.
A *loop* is a closed curve parametrized by a periodic function $g : \mathbb{R} \to \mathbb{C}$. If the loop g has constant curvature $R > 0$ then g is a circle of radius R^{-1} (use Theorem 8.5.7 in [3]; see also Theorem 0.2 above).
An *embedded loop* is a closed curve without self intersections. If Γ is a compact, connected, 1-dimensional submanifold of $\mathbb{C}$ with empty boundary then Γ is an embedded loop. Conversely, if g is an embedded loop then its image $g \equiv g(\mathbb{R})$ is a 1-dimensional submanifold of $\mathbb{C}$. Hence, g is diffeomorphic to $\mathbb{S}^1$ ([3], Theorem 3.4.1).
Let $H^1((0,1),\mathbb{C})$ be the standard Sobolev space of complex-valued maps defined on the interval (0,1). An inner product on the function space

$$H_{per} := \left\{ u \in H^1((0,1),\mathbb{C}) \mid u(0) = u(1) \ \right\}$$

is given by

$$\langle u, v \rangle = \int_0^1 u'v' + \left(\int_0^1 u \right) \cdot \left(\int_0^1 v \right) .$$

Notice that the Hilbert space H_{per} contains the subspace $\mathbb{C}$ of constant functions. Its topological dual space is denoted by H^{-1}_{per}.

1 Existence

We start by recalling the main features of the variational approach to problem (0.1). For details we refer to [7] and to [13]. Let $k \in C^0(\mathbb{C})$ be a given bounded function, and set

$$m(z) = \int_0^1 k(sz)s\ ds\ , \quad m : \mathbb{C} \to \mathbb{R}\,.$$

For $u \in H_{per}$ we put

$$\mathcal{L}(u) = \left(\int_0^1 |u'|^2 \right)^{1/2} \quad , \quad \mathcal{A}_k(u) = \int_0^1 m(u)u \cdot (iu')\,.$$

The functional $\mathcal{A}_k(u)$ is well defined on H_{per} as $u \in L^\infty$ for any $u \in H_{per}$, by Sobolev embedding theorem. The real number $\mathcal{A}_k(u)$ measures the algebraic area enclosed by the curve u with respect to the weight k. Moreover, the following isoperimetric inequality holds:

$$4\pi|\mathcal{A}_k(u)| \le \|k\|_\infty \mathcal{L}(u)^2 \quad \text{for any } u \in H_{per}\,. \tag{1.1}$$

In particular, for any constant $k_\infty \neq 0$ it turns out that

$$\mathcal{A}_{k_\infty}(u) = \frac{k_\infty}{2} \int_0^1 u \cdot (iu')\,. \tag{1.2}$$

In the next lemma we point out some simple remarks that will be useful in the sequel.

Lemma 1.1 *Let $k \in C^1(\mathbb{C})$. Assume that $M_k = \sup_{z\in\mathbb{C}} |(\nabla k(z) \cdot z)\ z|$ is finite and that there exists $\lim_{|z|\to\infty} k(z) = k_\infty \in \mathbb{R}$. Then*

$$|(k(z) - k_\infty)z| \le M_k \tag{1.3}$$

$$|(2m(z) - k(z))z| \le M_k \tag{1.4}$$

$$2\left|\mathcal{A}_{k-k_\infty}(u)\right| \le M_k\ \mathcal{L}(u) \quad \textit{for any } u \in H_{per}\,. \tag{1.5}$$

Proof. Inequality (1.3) follows from $k_\infty - k(z) = \int_1^\infty \nabla k(tz) \cdot z\ dt$. To check (1.4) use

$$2m(z) - k(z) = 2\int_0^1 (k(sz) - k(z))s\ ds = -2\int_0^1 s\ ds \int_s^1 \nabla k(tu) \cdot u\ dt\,.$$

Hölder inequality and (1.3) readily lead to (1.5), since $2\mathcal{A}_{k-k_\infty}(u) = \int_0^1 (2m(u) - k_\infty)u \cdot (iu')$.
□

The energy $\mathcal{E}_k : H_{per} \to \mathbb{R}$ is defined by

$$\mathcal{E}_k(u) = \mathcal{L}(u) + \mathcal{A}_k(u) = \left(\int_0^1 |u'|^2 \right)^{1/2} + \int_0^1 m(u) u \cdot (iu') \ .$$

If k is of class C^1 then the functional $\mathcal{E}_k$ is Frechét differentiable on $H_{per} \setminus \mathbb{C}$ and

$$\mathcal{E}_k'(u) \cdot \varphi = \frac{1}{\mathcal{L}(u)} \int_0^1 u' \cdot \varphi' + \int_0^1 k(u) \varphi \cdot (iu') \quad \text{for any } u \in H_{per} \setminus \mathbb{C},\ \varphi \in H_{per} \tag{1.6}$$

(see for example [7]). In particular, any critical point for $\mathcal{E}_k$ on $H_{per} \setminus \mathbb{C}$ parametrizes a smooth k-loop.

Remark 1.2 *Let k be as in Lemma 1.1, and assume in addition that $k_\infty \neq 0$. Then the energy $\mathcal{E}_k$ is unbounded from below. For, fix any map $u \in H_{per}$ such that $\mathcal{A}_{k_\infty}(u) < 0$. Then, using (1.5), we get*

$$\mathcal{E}_k(su) = \mathcal{L}(su) + \mathcal{A}_{k-k_\infty}(su) + \mathcal{A}_{k_\infty}(u) \le (1 + M_k) s \mathcal{L}(u) + s^2 \mathcal{A}_{k_\infty}(u) \to -\infty \quad \text{as } s \to \infty.$$

Next we introduce the curvature function

$$f(z) := 2k(z) + (\nabla k(z) \cdot z) \ , \quad f : \mathbb{C} \to \mathbb{R} \, .$$

Since $\int_0^1 f(sz) s \, ds = k(z)$, then from (1.6) we infer

$$\mathcal{E}_k'(u) \cdot u = \mathcal{L}(u) + \int_0^1 k(u) u \cdot (iu') = \mathcal{E}_f(u) \, . \tag{1.7}$$

Thus, $\mathcal{E}_k'(u) \cdot u$ equals the energy of u with respect to the curvature f. If k is of class C^2, then, coherently with (1.6), the functional $\mathcal{E}_f$ is Frechét differentiable on $H_{per} \setminus \mathbb{C}$, and

$$\mathcal{E}_f'(u) \cdot \varphi = \frac{1}{\mathcal{L}(u)} \int_0^1 u' \cdot \varphi' + \int_0^1 (2k(u) + (\nabla k(u) \cdot u)) \, \varphi \cdot (iu') \tag{1.8}$$

for any $u \in H_{per} \setminus \mathbb{C}$, $\varphi \in H_{per}$. Lastly we introduce the Nehari manifold

$$\Sigma_k = \left\{ u \in H_{per} \mid \mathcal{L}(u) > 0 \ , \ \mathcal{E}_k'(u) \cdot u = \mathcal{E}_f(u) = 0 \ \right\}$$

and the infimum

$$\underline{c} := \inf_{\Sigma_k} \mathcal{E}_k \, . \tag{1.9}$$

Remark 1.3 *Assume that $k \in C^2(\mathbb{C})$ satisfies $M_k < 1$ and $k(z) \to k_\infty = 0$ as $|z| \to \infty$. Then $\mathcal{E}_k(u) \ge (1 - M_k)\mathcal{L}(u)$ and $\mathcal{E}_k'(u) \cdot u \ge (1 - M_k)\mathcal{L}(u)$ for any $u \in H_{per}$, by (1.5), (1.7) and (1.3). Thus the energy is coercive with respect to the seminorm $\mathcal{L}(u)$ and the manifold Σ_k is empty.*

From now on we assume that $k \in C^2(\mathbb{C})$ satisfies the assumptions in Theorem 1.13. We point out that the set Σ_k is not empty, smooth, and it is a natural constraint for $\mathcal{E}_k$.

Lemma 1.4 *a) Let $u \in H_{per} \setminus \mathbb{C}$. Then $u \in \Sigma_k$ if and only if $\mathcal{E}_k(u) = \sup_{s>0} \mathcal{E}_k(su)$.*

b) Σ_k is a not empty submanifold of H_{per} of class C^1.

c) $0 < \underline{c} \le \pi/k_\infty$.

d) Every critical point u for $\mathcal{E}_k$ on Σ_k solves $\mathcal{E}_k'(u) = 0$.

Proof. We start by noticing that

$$\mathcal{E}_f'(u) \cdot u = -\mathcal{L}(u) + \int_0^1 (\nabla k(u) \cdot u) u \cdot (iu') \quad \text{for any } u \in \Sigma_k \tag{1.10}$$

by (1.8). Thus from (k_1) we readily get

$$-(1 + M_k)\mathcal{L}(u) \le \mathcal{E}_f'(u) \cdot u \le (-1 + M_k)\mathcal{L}(u) < 0 \quad \text{for any } u \in \Sigma_k \ . \tag{1.11}$$

To prove $a)$ we fix any function u in $H_{per} \setminus \mathbb{C}$ such that $\mathcal{E}_k(u) = \sup_{s>0} \mathcal{E}_k(su)$. Then the derivative of the function $s \mapsto \mathcal{E}_k(su)$ vanishes at $s = 1$, that is, $u \in \Sigma_k$. Conversely, if $u \in \Sigma_k$ we put $F(s) := \mathcal{E}_k(su)$ for $s > 0$. We use (1.6), (1.10) and (1.11) to get

$$\begin{aligned} sF'(s) &= \mathcal{E}_f(su) \\ sF'(s) + s^2 F''(s) &= \mathcal{E}_f'(su) \cdot (su) \le (-1 + M_k) s^2 \mathcal{L}(u) \ . \end{aligned}$$

In particular, $s = 1$ is a critical point for F on $\{s > 0\}$ and every critical point for F is a local maximum for F. Thus the function F achieves its absolute maximum at $s = 1$, that is, $\mathcal{E}_k(u) = \sup_{s>0} \mathcal{E}_k(su)$. This completes the proof of $a)$.
We point out that from $a)$ and from Remark 1.2 it follows that Σ_k is not empty and

$$\underline{c} = \inf_{\substack{u \in H_{per} \\ \mathcal{A}_{k_\infty}(u) < 0}} \sup_{s>0} \mathcal{E}_k(su) \,. \tag{1.12}$$

Since $\mathcal{E}_f'(u) \ne 0$ for any $u \in \Sigma_k$ by (1.11), then the constraint Σ_k has a normal direction at every point. Thus claim $b)$ is proved, as $\mathcal{E}_f$ is continuously differentiable on $H_{per} \setminus \mathbb{C} \supset \Sigma_k$.
To check that $\underline{c}$ is positive we use the isoperimetric inequality (1.1). For any $u \in \Sigma_k$ we get $0 = \mathcal{E}_f(u) = \mathcal{L}(u) + \mathcal{A}_f(u) \ge \mathcal{L}(u) - \|f\|_\infty (4\pi)^{-1} \mathcal{L}(u)^2$. In particular

$$\mathcal{L}(u) \ge \frac{4\pi}{\|f\|_\infty} \quad \text{for any } u \in \Sigma_k \,. \tag{1.13}$$

As $\mathcal{E}_f(u) = 0$ for any $u \in \Sigma_k$, then from (1.7) we get

$$2\mathcal{E}_k(u) = \mathcal{L}(u) + \int_0^1 (2m(u) - k(u))\, u \cdot (iu') \quad \text{for any } u \in \Sigma_k \,. \tag{1.14}$$

Thus from (1.4) we infer

$$2\mathcal{E}_k(u) \geq (1 - M_k)\mathcal{L}(u) \quad \text{for any } u \in \Sigma_k\,, \tag{1.15}$$

that compared with (1.13) gives $\underline{c} > 0$, since $M_k < 1$. To prove that $\underline{c} \leq \pi/k_\infty$ we parametrize the unit circle around 0 with the curve $\omega(t) = e^{2\pi i t}$ as in (0.3). By (1.12) it turns out that

$$\underline{c} \leq \sup_{s>0} \mathcal{E}_k(s(\omega + p)) = \sup_{s>0} \left[\mathcal{E}_{k_\infty}(s(\omega + p)) + \mathcal{A}_{k-k_\infty}(s(\omega + p))\right]$$

for any $p \in \mathbb{C}$. It is easy to compute

$$\mathcal{E}_{k_\infty}(s(\omega + p)) = 2\pi s - \pi k_\infty s^2\,.$$

Since $s(\omega + p)$ parametrizes the circle of radius s around sp, then

$$|\mathcal{A}_{k-k_\infty}(\omega)| = \left| \int_{D_s(sp)} (k(z) - k_\infty)\ dz \right| \leq M_k \int_{D_s(sp)} \frac{1}{|z|}\ dz$$

by the divergence theorem and by (1.3). Thus

$$|\mathcal{A}_{k-k_\infty}(\omega)| \leq M_k \int_{D_s(sp)} \frac{1}{s(|p| - 1)}\ dz = M_k \frac{\pi s}{|p| - 1}$$

and

$$\sup_{s>0} \mathcal{E}_k(s(\omega + p)) \leq \sup_{s>0}\ \pi s \left(2 + \frac{M_k}{|p| - 1} - k_\infty s\right) = \frac{\pi}{k_\infty} + O(|p|^{-1})\,. \tag{1.16}$$

Hence $\underline{c} \leq \pi/k_\infty$, as desired.
It remains to prove d). If $u \in \Sigma_k$ is a critical point for $\mathcal{E}_k$ on Σ_k then there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $\mathcal{E}_k'(u) \cdot \varphi = \lambda \mathcal{E}_f'(u) \cdot \varphi$ for any $\varphi \in H_{per}$. In particular $\lambda \mathcal{E}_f'(u) \cdot u = \mathcal{E}_k'(u) \cdot u = 0$ as $u \in \Sigma_k$. Thus $\lambda = 0$ by (1.11), and $\mathcal{E}_k'(u) = 0$. $\square$

Remark 1.5 *Argue as in the proof of claim a) to check that for any $p \in \mathbb{C}$ there exists a unique $s_p > 0$ such that $s_p(\omega + p) \in \Sigma_k$, where ω is defined in (0.3). As s_p is uniquely and implicitly defined by the equation $\mathcal{E}_f(s_p(\omega+p)) = 0$, and since the functional $\mathcal{E}_f$ is continuously differentiable on $H_{per} \setminus \mathbb{C}$, then the function $p \mapsto s_p$ is of class C^1 on $\mathbb{C}$ by (1.11).*

Remark 1.6 *Lemma 1.4 holds whenever $M_k < 1$ and $k(z) \to k_\infty > 0$ as $|z| \to \infty$.*

The next lemma concerns the compactness properties of the energy functional $\mathcal{E}_k$. We recall that $u_n \in \Sigma_k$ is a Palais Smale sequence for $\mathcal{E}_k$ on Σ_k at the level $c \in \mathbb{R}$ if there exists a sequence $\lambda_n \in \mathbb{R}$ such that

$$\mathcal{E}_k(u_n) \to c \tag{1.17}$$

$$\mathcal{E}_k'(u_n) - \lambda_n \mathcal{E}_f'(u_n) \to 0 \quad \textit{in } H_{per}^{-1}\,. \tag{1.18}$$

Lemma 1.7 *Let (u_n) be a Palais-Smale sequence for $\mathcal{E}_k$ on Σ_k at the level c. Then, up to a subsequence, either u_n converges in H_{per} to a k-loop u such that $\mathcal{E}_k(u) = c$, or $\mu := ck_\infty/\pi$ is a positive integer, and*

$$\left| \int_0^1 u_n \right| \to \infty \,, \quad u_n - \int_0^1 u_n \to U \quad \textit{in } H_{per} \,,$$

where $U(t) = k_\infty^{-1} \sigma e^{2\pi\mu it}$ for some $\sigma \in \mathbb{S}^1$.

Proof. Let $u_n \in \Sigma_k$ be a sequence satisfying (1.17) and (1.18) for some $c, \lambda_n \in \mathbb{R}$. Then the seminorms $\mathcal{L}(u_n)$ are uniformly bounded by (1.15). If the sequence $p_n := \int_0^1 u_n$ is bounded in $\mathbb{C}$ then we can assume that $u_n \rightharpoonup u$ weakly in H_{per}, for some $u \in H_{per}$. In particular, $u_n \to u$ uniformly on $[0,1]$ by Sobolev embedding theorem. Test (1.18) with u_n to get $0 = \mathcal{E}_k'(u_n) \cdot u_n = \lambda_n \mathcal{E}_f'(u_n) \cdot u_n + o(1)$. The sequence $\mathcal{E}_f'(u_n) \cdot u_n$ is bounded and bounded away from zero by (1.11). Thus $\lambda_n \to 0$, that implies $\mathcal{E}_k'(u_n) \to 0$ by (1.7). Since $|k(u_n)(u_n - u) \cdot (iu_n')| \le \|k\|_\infty \|u_n - u\|_\infty \, |u_n'| \to 0$ in L^1, then in particular

$$o(1) = \mathcal{E}_k'(u_n) \cdot (u_n - u) = \frac{1}{\mathcal{L}(u_n)} \int_0^1 u_n' \cdot (u_n' - u') + o(1) = \frac{1}{\mathcal{L}(u_n)} \int_0^1 |u_n' - u'|^2 + o(1) \,.$$

Taking (1.13) into account we infer that $u_n' \to u'$ strongly in L^2. Therefore $u_n \to u$ in the H_{per}-norm. Then $\mathcal{E}'(u) = 0$ and $\mathcal{E}_k(u) = c$ easily follow by continuity.
If the averages p_n are not bounded then, for a subsequence, it turns out that $u_n - p_n \rightharpoonup U$ weakly in H_{per} and uniformly on $[0,1]$, where $U \in H_{per}$ has zero mean value on $(0,1)$. We can also assume that there exists

$$\alpha := \lim_{n\to\infty} \mathcal{L}(u_n) \in \mathbb{R} \,.$$

Notice that $\alpha > 0$ by (1.13). Since $u_n - p_n \to U$ and $|u_n| \to \infty$ uniformly, then from (k$_1$) and (k$_2$) it follows that

$$(k(u_n) - k_\infty)u_n \to 0 \quad \text{uniformly} \tag{1.19}$$

$$k(u_n)(u_n - p_n) \to k_\infty U \quad \text{uniformly} \tag{1.20}$$

$$\nabla k(u_n) \cdot u_n \to 0 \quad \text{uniformly} \,. \tag{1.21}$$

Using (1.19) we get

$$\int_0^1 k(u_n) u_n \cdot (iu_n') = k_\infty \int_0^1 u_n \cdot (iu_n') + o(1) = k_\infty \int_0^1 U \cdot (iU') + o(1) \,,$$

as $\int_0^1 (iu_n') = 0$. In particular, from $\mathcal{E}_k'(u_n) \cdot u_n = 0$ and from (1.7) we infer that $U \neq 0$ and

$$\alpha = -k_\infty \int_0^1 U \cdot (iU') \,. \tag{1.22}$$

Next we notice that $(k(u_n)-k_\infty)p_n = (k(u_n)-k_\infty)(p_n-u_n)+(k(u_n)-k_\infty)u_n \to 0$ uniformly, since $k(u_n)\to k_\infty$ uniformly, $\sup_n \|u_n-p_n\|<\infty$ and by (1.19). This implies

$$\int_0^1 k(u_n)p_n\cdot(iu_n') = k_\infty p_n\cdot\int_0^1 (iu_n') + o(1) = o(1)\,.$$

Thus

$$\mathcal{E}_k'(u_n)\cdot(u_n-p_n) = -\mathcal{E}_k'(u_n)\cdot p_n = \int_0^1 k(u_n)p_n\cdot(iu_n') = o(1)\,,$$

as $u_n\in\Sigma_k$. With similar arguments, using (1.8) we obtain also

$$\mathcal{E}_f'(u_n)\cdot(u_n-p_n) = \alpha + 2k_\infty\int_0^1 U\cdot(iU') + o(1) = -\alpha + o(1) \tag{1.23}$$

by (1.20), (1.21) and (1.22). Consequently, from $\mathcal{E}_k'(u_n)\cdot(u_n-p_n) = \lambda_n\mathcal{E}_f'(u_n)\cdot(u_n-p_n)+o(1)$ it follows that $o(1)=\lambda_n(-\alpha+o(1))$. Since $\alpha>0$ then $\lambda_n\to 0$ and thus $\mathcal{E}_k'(u_n)\cdot\varphi\to 0$ for any $\varphi\in H_{per}$. Therefore

$$o(1) = \mathcal{E}_k'(u_n)\cdot\varphi = \frac{1}{\alpha}\int_0^1 U'\varphi' + 2k_\infty\int_0^1 \varphi\cdot(iU') + o(1) \quad \text{for any } \varphi\in H_{per}.$$

Thus, U solves $U''=\alpha k_\infty(iU')$, and in particular $\mathcal{L}(U)=\alpha=\lim_n\mathcal{L}(u_n)$ by (1.22). This is sufficient to conclude that $u_n-p_n\to U$ strongly in H_{per}. In addition, U is a non constant solution to the linear ordinary differential system $U''=\mathcal{L}(U)k_\infty(iU')$ with zero mean value on $(0,1)$. Therefore there exist an integer $\mu\ge 1$ and a point $\sigma\in\mathbb{S}^1$ such that

$$U(t) = \frac{1}{k_\infty}\ \sigma e^{2\pi\mu i t}\ ,\quad \mathcal{E}_{k_\infty}(U) = \frac{\pi\mu}{k_\infty}\,.$$

It remains to prove that $c=\pi\mu/k_\infty$. Notice that

$$(2m(u_n)-k_\infty)\,(u_n) = 2\int_0^1\ (k(su_n)-k_\infty)(su_n)\ ds \to 0$$

pointwise in $(0,1)$ by (k$_2$), (1.3) and by Lebesgue's Theorem. Since in addition $iu_n'\to iU'$ in L^2, then

$$2\mathcal{A}_{k-k_\infty}(u_n) = \int_0^1 (2m(u_n)-k_\infty)\,(u_n)\cdot(iu_n') = o(1)$$

by (1.4) and again by Lebesgue's Theorem. In conclusion, we get

$$\mathcal{E}_k(u_n) = \mathcal{L}(u_n) + \frac{k_\infty}{2}\int_0^1 (u_n-p_n)\cdot(iu_n') + \mathcal{A}_{k-k_\infty}(u_n) = \mathcal{E}_{k_\infty}(U)+o(1) = \frac{\pi\mu}{k_\infty}+o(1)\,.$$

The Lemma is completely proved. $\square$

Remark 1.8 *The Palais Smale condition fails if* $\underline{c} = \pi\mu/k_\infty$ *for some integer* $\mu \geq 1$. *For, take any sequence* $p_n \in \mathbb{C}$ *such that* $|p_n| \to \infty$ *and let* s_{p_n} *be the unique positive number such that* $s_{p_n}(\omega + p_n) \in \Sigma_k$, *as in Remark 1.5. Put* $u_n(t) = s_{p_n}(\omega(\mu t) + p_n)$ *and notice that the sequence* u_n *is unbounded and not relatively compact in* H_{per} *by (1.25) in the next subsection. On the other hand,* $u_n \in \Sigma_k$ *is a Palais Smale sequence for* $\mathcal{E}_k$ *on* Σ_k *at the level* $\pi\mu/k_\infty$ *(use (1.30) and (1.29) below).*

1.1 Existence of a minimal k-loop

Assume that k satisfies (k_1), (k_2) and define the energy level $\underline{c}$ as in (1.9). If $\underline{c} < \pi/k_\infty$, then Ekeland's variational principle and Lemma 1.7 guarantee that the minimal energy $\underline{c}$ is achieved by some $\underline{u} \in \Sigma_k$. Thus Lemma 1.4 immediately leads to our first existence result.

Theorem 1.9 *If* $k \in C^2(\mathbb{C})$ *satisfies* (k_1), (k_2) *and if* $\underline{c} < \pi/k_\infty$, *then there exists a minimal* k*-loop.*

Remark 1.10 *We notice that the assumption on the sign of* k_∞ *is not restrictive. Indeed,* $t \mapsto u(t)$ *is a* k*-loop if and only if* $t \mapsto u(1-t)$ *has curvature* $-k$.

Some conditions to ensure that $\underline{c} < \pi/k_\infty$ can be easily given. For instance, notice that

$$\underline{c} \leq \sup_{s>0} \mathcal{E}_k(s(\omega + p))$$

for any $p \in \mathbb{C}$, by (1.12), where ω is defined in (0.3). Since $s(\omega + p)$ parametrizes the circle of radius s around sp, then $\mathcal{A}_k(s(\omega + p)) = -\int_{D_s(sp)} k(z)\, dz$ by the divergence theorem. Thus

$$\mathcal{E}_k(s(\omega + p)) = 2\pi s - \int_{D_s(sp)} k(z)\, dz\,. \tag{1.24}$$

In particular, $\underline{c} < \pi/k_\infty$ if there exists a point $p \in \mathbb{C}$ such that

$$\sup_{s>0} \left(2\pi s - \int_{D_s(sp)} k(z)\, dz \right) < \frac{\pi}{k_\infty}\,.$$

This happens, for instance, if $k(z) > k_\infty$ on $\mathbb{C}$ (take $p = 0$). Actually weaker sufficient conditions can be given.

Theorem 1.11 *Assume that* $k \in C^2(\mathbb{C})$ *satisfies* (k_1), (k_2). *If* $k(z) > k_\infty$ *for* $|z|$ *large then there exists a minimal* k*-loop.*

Proof. We only have to show that $\underline{c} < \pi/k_\infty$. Put as before $\omega(t) = e^{2\pi i t}$. For any $p \in \mathbb{C}$ let $s_p > 0$ be the unique positive number defined in Remark 1.5. Then $\underline{c} \leq \mathcal{E}_k(s_p(\omega + p))$, since $s_p(\omega + p) \in \Sigma_k$.

Notice that $\mathcal{E}_f(u) = \mathcal{L}(u) + k_\infty \int_0^1 u \cdot (iu') + \int_0^1 (k(u) - k_\infty) \cdot (iu')$ for any $u \in H_{per}$, and therefore

$$0 = \mathcal{E}_f(s_p(\omega + p)) = 2\pi s_p - 2\pi k_\infty s_p^2 + s_p \int_0^1 (k(s_p(\omega + p)) - k_\infty) s_p(\omega + p) \cdot (i\omega')\,.$$

In particular we infer

$$k_\infty s_p = 1 - \int_0^1 (k(s_p(\omega + p)) - k_\infty) s_p(\omega + p) \cdot \omega \tag{1.25}$$

since $i\omega' = -2\pi\omega$. From (1.3) and (1.25) we get

$$1 - M_k \le k_\infty s_p \le 1 + M_k \quad \text{for any } p \in \mathbb{C}. \tag{1.26}$$

Next we take a sequence of points $p_n \in \mathbb{C}$ such that $|p_n| \to \infty$. By (1.24), (1.26), and since $k(z) > k_\infty$ for $|z|$ large enough we get

$$\underline{c} \le \mathcal{E}_k(s_{p_n}(\omega + p_n)) = \ 2\pi \ s_{p_n} - \int_{D_{s_{p_n}}(s_{p_n} p_n)} k(z) \ dz < \ 2\pi \ s_{p_n} - \pi k_\infty s_{p_n}^2\,.$$

Thus $\underline{c} < \sup\limits_{s>0} \left(2\pi s - \pi k_\infty s^2\right) = \pi / k_\infty$ and the theorem is completely proved. $\square$

Remark 1.12 *A slightly different proof can be obtained by following the arguments in [8], proof of Corollary 2.13. As in [8] the assumption on the sign of $k(z) - k_\infty$ can be weakened. It is sufficient to ask that there exist $\sigma \in \mathbb{S}^1$, $R, \delta > 0$ such that $k(z) > k_\infty$ for any $z \in \mathbb{C} \setminus D_R$ with $\left|\sigma - z|z|^{-1}\right| < \delta$.*

1.2 Existence of highly unstable k-loops

Here we prove the following existence result, that corresponds to Theorem 1.1 in [11].

Theorem 1.13 *Let $k \in C^2(\mathbb{C})$ be a given function satisfying* (k_1) *and* (k_2). *Then a k-loop exists, provided that*

$$c^* := \sup_{\substack{p \in \mathbb{C} \\ s > 0}} \left[2\pi s - \int_{D_s(p)} k(z) \ dz \right] < \frac{2\pi}{k_\infty}\,. \tag{1.27}$$

Proof. Define the Nehari manifold Σ_k and the minimal energy $\underline{c}$ as before. We can assume that

$$\underline{c} = \frac{\pi}{k_\infty} \quad \textit{and } \underline{c} \textit{ is not achieved}\,, \tag{1.28}$$

otherwise we are done, by Lemma 1.4 and by Theorem 1.9. We start by pointing out some remarks concerning the function $\omega(t) := \ e^{2\pi i t}$ already defined in (0.3). For any $p \in \mathbb{C}$ define $s_p > 0$ as in Remark 1.5. Notice that for any $s > 0$ it results $\mathcal{E}_k(s_p(\omega + p)) \ge \underline{c}$ by *a*) of Lemma 1.4. Thus, from (1.28) and (1.16) it follows that

$$\mathcal{E}_k(s_p(\omega + p)) = \frac{\pi}{k_\infty} + o(1) \quad \textit{as } |p| \to \infty\,. \tag{1.29}$$

Passing to the limit in (1.25) and using (k$_2$) we easily get

$$s_p = \frac{1}{k_\infty} + o(1) \quad as\ |p| \to \infty\,. \tag{1.30}$$

We claim that we can fix $\varepsilon > 0$ small enough, in such a way that

$$u \in \Sigma_k\ , \quad \mathcal{E}_k(u) \le \underline{c} + \varepsilon \quad \Rightarrow \quad \left| \int_0^1 u \right| > 1\,. \tag{1.31}$$

We argue by contradiction. Assume that there exist a sequence $u_n \in \Sigma_k$ such that $\mathcal{E}_k(u_n) = \underline{c} + o(1)$ and $\left|\int_0^1 u_n\right| \le 1$. By Ekeland's variational principle there exists a Palais Smale sequence $v_n \in \Sigma_k$ for $\mathcal{E}_k$ on Σ_k such that $\mathcal{E}_k(v_n) = \underline{c} + o(1)$ and $v_n - u_n \to 0$ in H_{per}. Since $\underline{c}$ is not achieved then the sequence v_n is not relatively compact. Thus, up to a subsequence we have that $\left|\int_0^1 v_n\right| \to \infty$, by Lemma 1.7. But then $\left|\int_0^1 u_n\right| \to \infty$ as well, a contradiction. Thus the claim is proved.
Next use (1.29) and (1.30) to fix a large radius R in such a way that

$$\mathcal{E}_k(s_p(\omega + p)) < \underline{c} + \varepsilon \quad \text{if} \ \ |p| = R \tag{1.32}$$

$$s_p > \frac{1}{2k_\infty} \quad \text{if } |p| = R\,. \tag{1.33}$$

Finally, we set

$$\overline{c} := \inf_{\phi \in \Phi} \ \sup_{z \in \overline{D}_R} \mathcal{E}_k(\phi(z))\,,$$

where $\overline{D}_R$ is the closure of the disk $D_R = \{\ |z| < R\ \}$, and

$$\Phi = \left\{ \phi \in C^0(\overline{D}_R, \Sigma_k) \mid \phi(p) = s_p(\omega + p) \text{ if } \ |p| = R \ \right\}\,.$$

Let ϕ be any map in Φ. Since $\displaystyle\int_0^1 \phi(z)\ dt = s_p \int_0^1 (\omega + p)\ dt = s_p p$ for any $p \in \partial D_R$, then by (1.33) and by degree arguments there exists a point $p_\phi \in D_R$ such that $\int_0^1 \phi(p_\phi) = 0$. Therefore

$$\sup_{p \in \overline{D}_1} \mathcal{E}_k(\phi(p)) \ge E_k(\phi(p_\phi)) > \underline{c} + \varepsilon$$

by (1.31). This proves that $\underline{c} + \varepsilon \le \overline{c}$. Standard variational arguments and (1.32) guarantee the existence of a Palais Smale sequence u_n for $\mathcal{E}_k$ on Σ_k at the level $\overline{c} > \underline{c} = \pi/k_\infty$. Notice that $\overline{c}$ is not an integer of π/k_∞, as $\underline{c} = \pi/k_\infty < \overline{c} \le c^*$ and since $c^* < 2\pi/k_\infty$ by assumption. Therefore u_n strongly converges to a k-loop by Lemma 1.7. □

As in [11] we can exhibit some sufficient conditions for (1.27) that lead us to the following existence result.

Proposition 1.14 *Let $k \in C^2(\mathbb{C})$ be a function satisfying* (k_1) *and* (k_2). *Then a k-loop exists, provided that one of the following conditions holds true:*

$$2 \inf_{p\in\mathbb{C}} k(p) > k_\infty \tag{1.34}$$

$$M_k < 2(\sqrt{2}-1)\,. \tag{1.35}$$

Proof. Put $k_0 = \inf_{p\in\mathbb{C}} k(p)$ and observe that

$$2\pi s - \int_{D_s(p)} k(z)\,dz \le 2\pi s - \pi k_0 s^2 \le \frac{\pi}{k_0}$$

for any $p \in \mathbb{C}$, $s > 0$. Therefore (1.27) follows from (1.34). Now we assume that (1.35) holds. We compute

$$\int_{D_s(p)} k(z)\,dz = 2\pi k_\infty s^2 - \mathcal{A}_{k-k_\infty}(s\omega + p)$$

and we use (1.5) to evaluate $|2\mathcal{A}_{k-k_\infty}(s(\omega+p))| \le M_k \mathcal{L}(s\omega) = 2\pi M_k s$. Thus

$$2\pi s - \int_{D_s(p)} k(z)\,dz \le \pi(2+M_k)s - \pi k_\infty s^2 \le \frac{\pi}{4k_\infty}\,(2+M_k)^2 < \frac{2\pi}{k_\infty}$$

for any $p \in \mathbb{C}$, $s > 0$. This implies that (1.27) holds. □

Other sufficient conditions involve the negative part $h_-(z) = -\min\{k(z) - k_\infty, 0\}$ of the difference $h(z) := k(z) - k_\infty$.

Proposition 1.15 *Let $k \in C^2(\mathbb{C})$ be a function satisfying* (k_1) *and* (k_2). *Then a k-loop exists, provided that one of the following conditions holds true:*

$$2\|h_-\|_\infty < 3k_\infty \tag{1.36}$$

$$h_- \in L^1(\mathbb{C}) \quad \textit{and} \quad \int_{\mathbb{C}} h_-\ dz < \pi/k_\infty \tag{1.37}$$

$$h_- \in L^2(\mathbb{C}) \quad \textit{and} \quad \int_{\mathbb{C}} h_-^2\ dz < 4\pi(\sqrt{2}-1)^2\,. \tag{1.38}$$

Proof. Condition (1.36) is equivalent to (1.34), as $\inf k = k_\infty - \|h_-\|_\infty$. Assume that (1.37) holds. Since $k = k_\infty + (k - k_\infty) \ge k_\infty - h_-$, then

$$2\pi s - \int_{D_s(p)} k(z)\,dz \le 2\pi s - \pi k_\infty s^2 + \int_C h_-(z)\,dz \le \frac{\pi}{k_\infty} + \int_{\mathbb{C}} h_-(z)\,dz$$

and (1.27) readily follows. Finally, if $h_- \in L^2$ we use Hölder inequality to estimate

$$2\pi s - \int_{D_s(p)} k(z)\,dz \le 2\pi s - \pi k_\infty s^2 + \int_{D_s(p)} h_-(z)\,dz \le -\pi k_\infty s^2 + 2\pi s + \left(\pi \int_C h_-^2\right)^{1/2} s\,.$$

Thus (1.38) implies (1.27). □

We can also state some existence results for $(k_\infty + h)$-loops, where $k_\infty > 0$ and $h : \mathbb{C} \to \mathbb{R}$ are given. The following corollary is an immediate consequence of Proposition 1.15.

Corollary 1.16 *Let $h \in C^2(\mathbb{C})$ be a mapping satisfying*

$$M_h := \sup_{p\in\mathbb{C}} |(\nabla h(z) \cdot z)\ z| < 1 \quad \text{and} \quad h(z) = o(|z|^{-1}) \quad \text{as } |z| \to \infty.$$

1. *A $(k_\infty + h)$-loop exists if $k_\infty > 0$ is large enough.*
2. *If $h_- \in L^1(\mathbb{C})$, then a $(k_\infty + h)$-loop exists if $k_\infty > 0$ is small enough.*
3. *If (1.38) is satisfied then for any $k_\infty > 0$ there exists a $(k_\infty + h)$-loop.*

2 Breaking symmetry

In this section we identify the circle $C_R = \{|z| = R\}$ with its parametrization $t \mapsto Re^{2\pi it}$. With this notation we have that C_R is a k-loop if and only if $Rk(\cdot) \equiv 1$ on C_R.
Theorem 0.1 is an immediate consequence of the next result.

Theorem 2.1 *Let $k \in C^2(\mathbb{C})$ be a curvature satisfying* (k$_1$) *and* (k$_2$). *Assume that there exists $R > 0$ such that $Rk(z) \equiv 1$ and $\nabla k(z) \cdot z \geq 0$ for any $z \in C_R$. If the circle C_R is a minimal k-loop, then $\nabla k(z) \cdot z \equiv 0$ on C_R.*

Proof. We start by noticing that $u \in H_{per}$ is a k-loop if and only if $u_R := Ru$ is a k_R-loop, where $k_R(z) = R^{-1}k(R^{-1}z)$. Therefore we are allowed to assume $R = 1$ and $C_R = \mathbb{S}^1$. Hence $k \equiv 1$ on the unit sphere, $\mathbb{S}^1$ is a minimal k-loop and

$$A(z) := \nabla k(z) \cdot z \geq 0 \quad \text{on } \mathbb{S}^1. \tag{2.1}$$

Since $k \in C^2(\mathbb{C})$, then the energy $\mathcal{E}_k$ is twice differentiable on $H_{per} \setminus \mathbb{C}$, and

$$\begin{aligned}\mathcal{E}_k''(u)[\varphi, \psi] = \frac{1}{\mathcal{L}(u)} \int_0^1 \varphi' \cdot \psi' \quad &- \quad \frac{1}{\mathcal{L}(u)^3} \left(\int_0^1 u' \cdot \varphi'\right) \left(\int_0^1 u' \cdot \psi'\right) \\ &+ \quad \int_0^1 \varphi \cdot \left[k(u)(i\psi') + (\nabla k(u) \cdot \psi)(iu')\right]\end{aligned}$$

for all $u \in H_{per} \setminus \mathbb{C}$, and for any $\varphi, \psi \in H_{per}$ (see [7]). To compute $\mathcal{E}_k''(\omega)$ we notice that $\mathcal{L}(\omega) = 2\pi$ and $\omega'' = 2\pi(i\omega') = -(2\pi)^2\omega$. In particular $\int \omega' \cdot \varphi' = (2\pi)^2 \int \omega \cdot \varphi$ for any $\varphi \in H_{per}$. Since k is constant on $\mathbb{S}^1$ then $\nabla k(\omega) = A(\omega)\omega$, where A is defined in (2.1). Thus we get

$$\begin{aligned}\mathcal{E}_k''(\omega)[\varphi, \psi] = \frac{1}{2\pi} \int_0^1 \varphi' \cdot \psi' \quad &- \quad 2\pi \left(\int_0^1 \omega \cdot \varphi\right) \left(\int_0^1 \omega \cdot \psi\right) \\ &+ \quad \int_0^1 \varphi \cdot \left[(i\psi') - 2\pi A(\omega)(\omega \cdot \psi)\omega\right]\end{aligned} \tag{2.2}$$

for any $\varphi, \psi \in H_{per}$. From (2.2) we firstly get $\mathcal{E}_k''(\omega)[\varphi, \omega] = -2\pi \int_0^1 (1 + A(\omega))\, \omega \cdot \varphi$ for any $\varphi \in H_{per}$, and in particular

$$\mathcal{E}_k''(\omega)[\omega, \omega] = -2\pi \left(1 + \int_0^1 A(\omega)\right) < 0\,. \tag{2.3}$$

Taking instead $\varphi \equiv p$ to be a constant function we get

$$\mathcal{E}_k''(\omega)[p, \omega] = -2\pi \int_0^1 A(\omega)\omega \cdot p\,, \tag{2.4}$$

since ω has zero mean value on $(0, 1)$. From (2.2) we infer also

$$\mathcal{E}_k''(\omega)[p, p] = -2\pi \int_0^1 A(\omega)(\omega \cdot p)^2\,. \tag{2.5}$$

For any point $p \in \mathbb{C}$ we put $s(t) := s_{tp}$, where $s_{tp} > 0$ is defined by the condition $s_{tp}(\omega + tp) \in \Sigma_k$, as in Remark 1.5. Since ω is a minimal k-loop then the function $g(t) := \mathcal{E}_k(s(t)(\omega + tp))$ attains its minimum at $t = 0$. Notice that $s(0) = 1$, since ω is a k-loop. Moreover the map $s(t)$ is of class C^1 on $(0, \infty)$ by Remark 1.5 and

$$\mathcal{E}_k'(s(t)(\omega + tp)) \cdot (\omega + tp) = 0 \quad \text{for any } t \in \mathbb{R}. \tag{2.6}$$

We compute

$$g'(t) = \mathcal{E}_k'(s(t)(\omega + tp)) \cdot \big(s'(t)(\omega + tp) + s(t)p\big) = s(t)\mathcal{E}_k'((s(t)(\omega + tp)) \cdot p\,,$$

by (2.6). Thus g is twice differentiable, and

$$g''(t) = s'(t)\mathcal{E}_k'((s(t)(\omega + tp)) \cdot p + s(t)\mathcal{E}_k''((s(t)(\omega + tp))\,\big[p, s'(t)(\omega + tp) + p\big]\,.$$

From $s(0) = 1$ and $\mathcal{E}_k'(\omega) = 0$ we get

$$g''(0) = \beta\mathcal{E}_k''(\omega)[p, \omega] + \mathcal{E}_k''(\omega)[p, p]\,, \tag{2.7}$$

where we have set $\beta = s'(0)$. To compute β we differentiate (2.6) with respect to t:

$$\mathcal{E}_k'((s(t)(\omega + tp)) \cdot p + \mathcal{E}_k''((s(t)(\omega + tp))[\omega + tp, s'(t)(\omega + tp) + s(t)p] = 0\,.$$

Thus at $t = 0$ it holds that $\mathcal{E}_k''(\omega)[\omega, \beta\omega + p] = 0$, that compared with (2.7) gives

$$g(0) = -\,\frac{(\mathcal{E}_k''(\omega)[p, \omega])^2}{\mathcal{E}_k''(\omega)[\omega, \omega} \; + \; \mathcal{E}_k''(\omega)[p, p] = 2\pi \left[\frac{\left(\int_0^1 A(\omega)(\omega \cdot p)\right)^2}{1 + \int_0^1 A(\omega)} - \int_0^1 A(\omega)(\omega \cdot p)^2\right]$$

by (2.3), (2.4) and (2.5). Since 0 is a minimum point for g then $g''(0) \geq 0$, that is,

$$\begin{aligned} \left(\int_0^1 A(\omega)(\omega\cdot p)^2\right)\left(1+\int_0^1 A(\omega)\right) &\leq \left(\int_0^1 A(\omega)(\omega\cdot p)\right)^2 \\ &\leq \left(\int_0^1 A(\omega)(\omega\cdot p)^2\right)\left(\int_0^1 A(\omega)\right) \end{aligned}$$

by Hölder inequality. We infer that $\int A(\omega)(\omega\cdot p)^2 = 0$ for any $p \in \mathbb{C}$, since $A(\omega) \geq 0$ by assumption. Take $p = 1 \in \mathbb{C}$ and then $p = i$ to get

$$0 = \int_0^1 A(\omega)[(\omega\cdot 1)^2 + (\omega\cdot i)^2] = \int_0^1 A(\omega)\,.$$

Therefore $A(\omega) \equiv 0$, and the theorem is proved. □

3 Multiplicity and non existence

From now on we restrict our attention to radially symmetric curvatures $k(P) \equiv k(|P|)$. Quite trivially, if k is any continuous radially symmetric map on $\mathbb{C}$ such that $k(z) \to k_\infty > 0$ as $|z| \to \infty$, then there exists $R > 0$ such that $Rk(R) = 1$. In this case the circle C_R is a k-loop. In addition, if u is a k-loop then $\mathcal{R}\circ u$ is a k-loop for any rotation $\mathcal{R}$ of the complex plane. Therefore, the next multiplicity result immediately follows from Theorem 0.1.

Corollary 3.1 *Let $k \in C^2(\mathbb{C})$ be a radially symmetric function satisfying* (k$_1$), (k$_2$). *Assume that $k'(R) > 0$ for any R such that $Rk(R) = 1$. If $\underline{c}$ is achieved then no minimal k-loop is a circle around the origin. Hence there exist at least one round k-loop and a rotationally-invariant family of not round k-loops.*

It has to be noticed that the class of curvatures described in Corollary 3.1 is not empty. For instance, let $\lambda > 0$ and put

$$k(r) = 1 + \lambda\,\frac{r^2-1}{r^4+1}\,.$$

If λ is sufficiently close to 0 then k satisfies (k$_1$) and (k$_2$). In addition $Rk(R) = 1$ if and only if $R = 1$, and $k'(1) > 0$. The existence of a minimal k-loop is given by Theorem 1.11.

Multiple k loops may exist also in a more general setting, as the next example shows.

Example 3.2 *There exist a positive, radially symmetric and increasing curvature $k : \mathbb{C} \to \mathbb{R}$ with $M_k > 1$ that has a round k-loop and a not round embedded k-loop.*

Proof. Let $a < b$ be a pair of positive numbers. Let $k_{a,b}$ be any smooth function such that

$$k_{a,b}(z) := \frac{ab}{(a^2+b^2-|z|^2)^{3/2}} \tag{3.1}$$

on the disk $\{|z| \leq b\}$. Notice that $k_{a,b}$ can be taken to be radially symmetric and strictly increasing as a function of the distance from the origin.
The ellipse $E_{a,b}$ of equation $a^{-2}x^2 + b^{-2}y^2 = 1$ has curvature k at any point. For, we parametrize $E_{a,b}$ by the curve $g(t) = (a \cos t, b \sin t)$. Accordingly with (0.4), for any $t \in \mathbb{R}$ the curvature of $E_{a,b}$ at the point $g(t)$ is given by

$$K_{g(t)} = \frac{ab}{\left(a^2 \sin^2 t + b^2 \cos^2 t\right)^{3/2}} .$$

Since $a^2 \sin^2 t + b^2 \cos^2 = a^2 + b^2 - |g(t)|^2$, then $K_{g(t)} = k_{a,b}(g(t))$ and hence $E_{a,b}$ is a k-loop. By elementary continuity arguments there exists a radius $R \in (a, b)$ such that $Rk_{a,b}(z) \equiv 1$ on the circle C_R. Thus the circle C_R and the ellipse $E_{a,b}$ are distinct k-loops. □

In Corollary 3.1 and in Example 3.2 the existence of a radius $R > 0$ such that C_R is a k-loop was obtained via elementary continuity arguments. Indeed the existence of a round k-loop is a necessary condition for the existence of k-loops.

Proposition 3.3 *Let $k : \mathbb{C} \to \mathbb{R}$ be a continuous radially symmetric function. If there exists a k-loop, then there exists a circle C_R about the origin, such that C_R is a k-loop.*

Proof. Let $g : \mathbb{R} \to \mathbb{C}$ be a periodic parametrization by arclength of a k-loop. Then g solves

$$\begin{cases} g'' = k(|g|)(ig') \\ |g'| = 1 . \end{cases} \tag{3.2}$$

Let $P_0 = g(t_0)$ be a point on g such that $|P_0| = \bar{R} := \max\{|P| \mid P \in g\}$. Set $v(t) := |g(t)|^2$ and compute $v'' = 2(|g'|^2 + g \cdot g'') \geq 2(1 - |k(|g|)g|)$. Since v takes its maximum value at t_0, then $0 \geq v''(t_0) \geq 2(1 - |k(\bar{R})|\bar{R})$. Hence $\bar{R}|k(\bar{R})| \geq 1$. By the continuity of the function $r \mapsto r|k(r)|$, there exists a radius $R \in (0, \bar{R}]$ such that $R|k(R)| = 1$. Hence the circle C_R is a k-loop. □

We conclude this section with a nonexistence result that is an immediate consequence of Proposition 3.3.

Corollary 3.4 *Let $k : \mathbb{C} \to \mathbb{R}$ be a continuous radially symmetric function. If $R|k(R)| < 1$ for any R, then there are no k-loops.*

4 Uniqueness up to a similitude

In this section we prove Theorem 0.2. We start by fixing some notation. Let g be an embedded loop in $\mathbb{C}$, and let $k : g \to \mathbb{R}$ be its curvature. We assume that g is positively oriented, so that the interior is to the left. Let P_1, P_2 be two distinct points in g. We denote by $A_g(P_1, P_2)$

the closed arc of g having endpoints P_1 and P_2, oriented accordingly to the orientation of g. The open arc will be denoted by $\mathring{A}_g\,(P_1, P_2)$.

Following Osserman's definition in [15], we let $C_R^g(X) = \{P \in \mathbb{C} \mid |P - X| = R\}$ to be the circumscribed circle about g. Thus, R is the minimum positive number r such that there exists a circle of radius r including g.

To prove Theorem 0.2 we need a preliminary result, which is based on Hopf's maximum principle. More precisely, the next Lemma is a consequence of the *Touching Lemma* for the mean curvature operator (see for example [16] and the monograph [17]). Although it is essentially well known, we state it here for sake of completeness.

Lemma 4.1 *Let $P \in g \cap C_R^g(X)$. If $k(M) \le 1/R$ for any $M \in g$ close to P, then $g \cap C_R^g(X)$ contains an arc around P.*

Proof. Up to a rotation and a translation we can assume that $X = 0$ and $P = iR$. An arc of $C_R(0)$ around P is the graph of the function $\gamma(t) = \sqrt{R^2 - t^2}$. Since g is an immersion then g is locally the graph of a function $f : I = (-\delta, \delta) \to \mathbb{R}$ such that $f(0) = R$. By assumption $k \le 1/R$ in a neighborhood of P, and therefore

$$\left(\frac{\gamma'}{\sqrt{1 + |\gamma'|^2}}\right)' + \frac{1}{R} = 0 \le \left(\frac{f'}{\sqrt{1 + |f'|^2}}\right)' + \frac{1}{R}$$

for any t close to 0. In addition we have that $f \le \gamma$ and $f(0) = \gamma(0) = R$. Thus $f \equiv \gamma$ in a neighborhood of 0, by Theorem 2.3 in [16]. $\square$

Proof of Theorem 0.2 Let k and g be as in the statement of the theorem. Then g is convex, since its curvature is positive (see for example [3], Theorem 9.6.2). Let $C_R^g(X)$ be the circumscribed circle about g. By Lemma 3 in [15] it turns out that $k(P) \ge 1/R$ for any $P \in g \cap C_R^g(X)$.

We claim that g coincides with its circumscribed circle. By contradiction, we assume that $g \cap C_R^g(X)$ is strictly contained in $C_R^g(X)$. We distinguish two cases, depending on the number of connected components in $C_R^g(X) \setminus g$.

Case 1: the set $C_R^g(X) \setminus g$ is not connected. Then we can fix two distinct points P_1 and P_2 on $g \cap C_R^g(X)$ such that the arcs $A_g(P_1, P_2)$ and $A_g(P_2, P_1)$ are not contained in the circle $C_R^g(X)$. By Lemma 4 in [15], there exist two points $Q_1 \in \mathring{A}_g\,(P_1, P_2)$ and $Q_2 \in \mathring{A}_g\,(P_2, P_1)$ such that

$$|Q_i - X| < R = |P_j - X|\ , \quad k(Q_i) < 1/R \le k(P_j)$$

for $i, j = 1, 2$. Since k is a non-increasing function of the distance from the origin, then $|Q_i| > |P_j|$. Thus $X \ne 0$ and

$$0 < |Q_i|^2 - |P_j|^2 = |Q_i - X|^2 - |P_j - X|^2 + 2X \cdot (Q_i - P_j) < 2X \cdot (Q_i - P_j)\,,$$

that is

$$\min\{X \cdot Q_1, X \cdot Q_2\} > \max\{X \cdot P_1, X \cdot P_2\}\,. \tag{4.1}$$

Denote by $[Q_1, Q_2]$ the segment joining Q_1 and Q_2, and by $[P_1, P_2]$ the segment joining P_1 and P_2. By the convexity of the curve g the segments $[Q_1, Q_2]$ and $[P_1, P_2]$ intersect at a point Q. Since $Q \in [Q_1, Q_2]$ then $X \cdot Q > \max\{X \cdot P_1, X \cdot P_2\}$ by (4.1). On the other hand, $X \cdot Q \le \max\{X \cdot P_1, X \cdot P_2\}$ as $Q \in [P_1, P_2]$, a contradiction. Thus case 1 is excluded.

Case 2: the set $C^g_R(X)\backslash g$ is connected. Then $C^g_R(X)\backslash g = \mathring{A}_g\,(P_2, P_1)$ and the arc $A_g(P_1, P_2)$ is contained in $C^g_R(X)$. In particular $k(P) = 1/R$ for any $P \in A_g(P_1, P_2)$. Since $C^g_R(X)$ is the smallest circle circumscribing g, then $A_g(P_1, P_2)$ is larger than a semicircle ([15], Lemma 2). In particular the antipodal $\tilde{P}_1 = 2X - P_1$ to P_1 belongs to $A_g(P_1, P_2)$. Up to a rotation and up to a change of indexes we can assume that P_1, $\tilde{P}_1$ lie on the same vertical line $\mathcal{V}$, with P_1 above $\tilde{P}_1$ and with $A_g(P_1, \tilde{P}_1)$ on the left of $\mathcal{V}$.
By Lemma 4 in [15], there exist a point $Q \in \mathring{A}_g\,(P_2, P_1)$ such that $k(Q) < 1/R$. Since g is convex, then Q lies on the right of the vertical line $\mathcal{V}$. By Lemma 4.1 there exists $M \in A_g(Q, P_1)$ on the right of $\mathcal{V}$ such that $k(M) > 1/R$. Since $k \equiv 1/R$ on $A_g(P_1, P_2)$ and since k is radially symmetric and non-increasing by assumption then

$$|M| < |P| < |Q| \quad \text{for any } P \in A_g(P_1, P_2) \subset C^g_R(X)\,. \tag{4.2}$$

In particular, $X \neq 0$, as $|X - Q| < R$. We put $P_0 := X(1 - R|X|^{-1}) \in C^g_R(X)$ and $\tilde{P}_0 := 2X - P_0 \in C^g_R(X)$.
We claim that 0 can not belong to $\mathbb{C} \setminus \overline{D_R(X)}$. Indeed, if $|X| \ge R$ then the point P_0 is the minimal distance projection of 0 on $\overline{D_R(X)}$. Since $|M| < |P|$ for any $P \in A_g(P_1, P_2)$ and since M is in the interior of $D_R(X)$, then $P_0 \in \mathring{A}_g\,(P_2, P_1) \subset \mathring{A}_g\,(\tilde{P}_1, P_1)$. But then $\tilde{P}_0 \in A_g(P_1, \tilde{P}_1)$. Thus $|\tilde{P}_0| < |Q|$, by (4.2). However this is impossible, as $|\tilde{P}_0| = |X| + R$ and $|Q| \le |Q - X| + |X| < R + |X|$.
Thus $0 \in D_R(X)$ and $|P_0| = R - |X| > 0$. Since $|P_1|, |\tilde{P}_1| < |Q|$ and since Q lies on the right of the axes $\mathcal{V}$ joining $P_1, \tilde{P}_1$, then 0 is on the left of $\mathcal{V}$. Therefore $P_0 \in A_g(P_1, \tilde{P}_1) \subset C^g_R(X)$ and $k(P_0) = 1/R$. Thus $k \equiv 1/R$ on $C_{|P_0|}(0)$ and $k \le 1/R$ outside $D_{|P_0|}(0)$, as k is radially symmetric and non increasing. Finally, we notice that the circle $C_{|P_0|}(0)$ is tangent to $C^g_R(X)$ at P_0 from the interior. Thus $k \le 1/R$ in a neighborhood of P_1 and therefore P_1 is in the interior of $g \cap C^g_R(X)$ by Lemma 4.1, a contradiction. The theorem is completely proved. $\square$

Theorem 0.2 provides a new characterization of circles.

Corollary 4.2 *Round circles are the only convex loops in $\mathbb{C}$ whose curvature is a non increasing function of the Euclidean distance from a fixed point.*

We conclude the paper by pointing out the following uniqueness (up to homothety) result, that is an immediate consequence of Theorem 0.2.

Corollary 4.3 *Let k be a continuous, positive and radially symmetric function. If k is radially decreasing then any embedded k-loop is a circle around the origin.*

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