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Interval Temporal Logic Model Checking: the Border Between Good and Bad HS Fragments

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Abstract. The model checking problem has thoroughly been explored in the context of standard point-based temporal logics, such as LTL, CTL, and CTL*, whereas model checking for interval temporal logics has been brought to the attention only very recently.
In this paper, we prove that the model checking problem for the logic of Allen's relations *started-by* and *finished-by* is highly intractable, as it can be proved to be **EXPSPACE**-hard. Such a lower bound immediately propagates to the full Halpern and Shoham's modal logic of time intervals (HS). In contrast, we show that other noteworthy HS fragments, namely, Propositional Neighbourhood Logic extended with modalities for the Allen relation *starts* (resp., *finishes*) and its inverse *started-by* (resp., *finished-by*), turn out to have—maybe unexpectedly—the same complexity as LTL (i.e., they are **PSPACE**-complete), thus joining the group of other already studied, well-behaved albeit less expressive, HS fragments.

1 Introduction

Model checking (MC) is one of the most successful techniques in the area of formal methods. It allows one to automatically check whether some desired properties of a system, specified by a temporal logic formula, hold over a model of it (generally, a Kripke structure). MC has proved itself to be extremely useful in formal verification [7], but it has also been successfully exploited in various areas of AI, ranging from planning to configuration and multi-agent systems (see, for instance, [9, 17]).

Point-based temporal logics, such as LTL [26], CTL, and CTL* [8], that allow one to predicate over computation states/worlds, are usually adopted in MC as the specification language, as they are suitable for practical purposes in many application domains. However, some relevant temporal properties, that involve, for instance, actions with duration, accomplishments, and temporal aggregations, are inherently "interval-based" and thus cannot be expressed by point-based logics. Here, we focus on MC algorithms for interval temporal logic (ITL).

ITLs take intervals, instead of points, as their primitive entities, providing an alternative setting for reasoning about time [11, 25, 31, 32]. They have been

applied in various areas of computer science and AI, including formal verification, computational linguistics, planning, and multi-agent systems [2, 14, 25, 27, 33]. In order to check interval properties of computations, one needs to collect information about states into computation stretches: each finite path of a Kripke structure is interpreted as an interval, whose labelling is defined on the basis of the labelling of the component states.

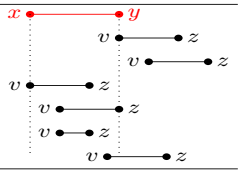
Halpern and Shoham's modal logic of time intervals HS [11] is the most famous among ITLs. It features one modality for each of the 13 possible ordering relations between pairs of intervals (the so-called Allen's relations [1]), apart from equality. The satisfiability problem for HS turns out to be highly undecidable for all relevant (classes of) linear orders [11]. The same holds for most fragments of it [3, 13, 18]. However, some meaningful exceptions exist, including the logic of temporal neighbourhood $\mathsf{A\overline{A}}$ and the logic of sub-intervals D [4–6, 24].

In this paper, we address some open issues in the MC problem for HS, which only recently entered the research agenda [14–16, 19–23]. In [19, 23], Montanari et al. deal with MC for full HS over Kripke structures (under the homogeneity assumption [28]). They introduce the problem and prove its non-elementary decidability and **PSPACE**-hardness. Since then, the attention was also brought to the fragments of HS, which, similarly to what happens with satisfiability, are often computationally better. Here, we focus on the border between good and bad HS fragments, showing the criticality of the combined use of modalities for interval prefixes and suffixes (modalities for Allen's relations *started-by* and *finished-by*). On the one hand, we prove that MC for the HS fragment BE, whose modalities can express properties of both prefixes and suffixes of intervals, is **EXPSPACE**-hard, and this lower bound immediately propagates to full HS. On the other hand, we show that the complexity of MC for HS fragments where properties of prefixes and suffixes of intervals are considered separately is markedly lower. In [22] the authors proved that if we consider only properties of future and past intervals, MC is in $\mathbf{P^{NP}}$; if modalities for interval extensions to the left and to the right are added, MC becomes **PSPACE**-complete [20]. Here we prove that MC for the HS fragment $\mathsf{A\overline{A}B\overline{B}}$ (resp., $\mathsf{A\overline{A}E\overline{E}}$), that allows one to express properties of interval prefixes (resp., suffixes), future and past intervals, and right (resp., left) interval extensions, is in **PSPACE**. Since MC for the HS fragment featuring only one modality for right (resp., left) interval extensions is **PSPACE**-hard [22], **PSPACE**-completeness immediately follows. Moreover, we show that if we restrict HS to modalities either for interval prefixes or for interval suffixes (HS fragments B and E), MC turns out to be **co-NP**-complete.

The MC problem for epistemic extensions of some HS fragments have been investigated by Lomuscio and Michaliszyn [14–16] (a detailed account of their results can be found in [19]). However, their semantic assumptions differ from those of [23], thus making it difficult to compare the two research lines.

In the next section, we introduce the fundamental elements of the MC problem for HS and its fragments. Then, in Section 3 we focus on the fragment BE, while in Section 4 we deal with $\mathsf{A\overline{A}E\overline{E}}$ and E (and with $\mathsf{A\overline{A}B\overline{B}}$ and B). Conclusions provide an assessment of the work done and outline future research directions.

Table 1. Allen's relations and corresponding HS modalities.

Allen relation	HS	Definition w.r.t. interval structures	Example
MEETS	$\langle A \rangle$	$[x,y]\mathcal{R}_A[v,z] \iff y = v$	
BEFORE	$\langle L \rangle$	$[x,y]\mathcal{R}_L[v,z] \iff y < v$	
STARTED-BY	$\langle B \rangle$	$[x,y]\mathcal{R}_B[v,z] \iff x = v \wedge z < y$	
FINISHED-BY	$\langle E \rangle$	$[x,y]\mathcal{R}_E[v,z] \iff y = z \wedge x < v$	
CONTAINS	$\langle D \rangle$	$[x,y]\mathcal{R}_D[v,z] \iff x < v \wedge z < y$	
OVERLAPS	$\langle O \rangle$	$[x,y]\mathcal{R}_O[v,z] \iff x < v < y < z$	

2 Preliminaries

The interval temporal logic HS. An interval algebra to reason about intervals and their relative order was proposed by Allen in [1], while a systematic logical study of interval representation and reasoning was done a few years later by Halpern and Shoham, who introduced the interval temporal logic HS featuring one modality for each Allen relation, but equality [11]. Table 1 depicts 6 of the 13 Allen's relations, together with the corresponding HS (existential) modalities. The other 7 relations are the 6 inverse relations (given a binary relation $\mathcal{R}$, the inverse relation $\overline{\mathcal{R}}$ is such that $b\overline{\mathcal{R}}a$ if and only if $a\mathcal{R}b$) and equality.

The HS language consists of a set of proposition letters $\mathcal{AP}$, the Boolean connectives $\neg$ and $\wedge$, and a temporal modality for each of the (non trivial) Allen's relations, i.e., $\langle A \rangle$, $\langle L \rangle$, $\langle B \rangle$, $\langle E \rangle$, $\langle D \rangle$, $\langle O \rangle$, $\langle \overline{A} \rangle$, $\langle \overline{L} \rangle$, $\langle \overline{B} \rangle$, $\langle \overline{E} \rangle$, $\langle \overline{D} \rangle$, and $\langle \overline{O} \rangle$. HS formulas are defined by the grammar $\psi ::= p \mid \neg\psi \mid \psi \wedge \psi \mid \langle X \rangle \psi \mid \langle \overline{X} \rangle \psi$, where $p \in \mathcal{AP}$ and $X \in \{A, L, B, E, D, O\}$. In the following, we will also exploit the standard logical connectives (disjunction $\vee$, implication $\rightarrow$, and double implication $\leftrightarrow$) as abbreviations. Furthermore, for any modality X, the dual universal modalities $[X]\psi$ and $[\overline{X}]\psi$ are defined as $\neg\langle X \rangle \neg\psi$ and $\neg\langle \overline{X} \rangle \neg\psi$, respectively.

Given any subset of Allen's relations $\{X_1, \cdots, X_n\}$, we denote by $\mathsf{X_1} \cdots \mathsf{X_n}$ the HS fragment featuring existential (and universal) modalities for $X_1, \ldots, X_n$ only.

W.l.o.g., we assume the *non-strict semantics of HS*, which admits intervals consisting of a single point[5]. Under such an assumption, all HS modalities can be expressed in terms of modalities $\langle B \rangle, \langle E \rangle, \langle \overline{B} \rangle$, and $\langle \overline{E} \rangle$ [31]. HS can thus be viewed as a multi-modal logic with these 4 primitive modalities and its semantics can be defined over a multi-modal Kripke structure, called *abstract interval model*, where intervals are treated as atomic objects and Allen's relations as binary relations between pairs of intervals. Since later we will focus on the HS fragments $\mathsf{A\overline{A}E\overline{E}}$ and $\mathsf{A\overline{A}B\overline{B}}$—which respectively do not feature $\langle B \rangle$, $\langle \overline{B} \rangle$ and $\langle E \rangle$, $\langle \overline{E} \rangle$—we add both $\langle A \rangle$ and $\langle \overline{A} \rangle$ to the considered set of HS modalities.

Definition 1. *[19] An* abstract interval model *is a tuple* $\mathcal{A} = (\mathcal{AP}, \mathbb{I}, A_{\mathbb{I}}, B_{\mathbb{I}}, E_{\mathbb{I}}, \sigma)$*, where* $\mathcal{AP}$ *is a set of proposition letters,* $\mathbb{I}$ *is a possibly infinite set of atomic objects (worlds),* $A_{\mathbb{I}}$*,* $B_{\mathbb{I}}$*, and* $E_{\mathbb{I}}$ *are three binary relations over* $\mathbb{I}$*, and* $\sigma : \mathbb{I} \mapsto 2^{\mathcal{AP}}$ *is a (total) labeling function, which assigns a set of proposition letters to each world.*

[5] All the results we prove in the paper hold for the strict semantics as well.

In the interval setting, $\mathbb{I}$ is interpreted as a set of intervals and $A_\mathbb{I}$, $B_\mathbb{I}$, and $E_\mathbb{I}$ as Allen's relations A (*meets*), B (*started-by*), and E (*finished-by*), respectively; σ assigns to each interval in $\mathbb{I}$ the set of proposition letters that hold over it.

Given an abstract interval model $\mathcal{A} = (\mathcal{AP}, \mathbb{I}, A_\mathbb{I}, B_\mathbb{I}, E_\mathbb{I}, \sigma)$ and an interval $I \in \mathbb{I}$, the truth of an HS formula over I is inductively defined as follows:

- $\mathcal{A}, I \models p$ iff $p \in \sigma(I)$, for any $p \in \mathcal{AP}$;
- $\mathcal{A}, I \models \neg\psi$ iff it is not true that $\mathcal{A}, I \models \psi$ (also denoted as $\mathcal{A}, I \not\models \psi$);
- $\mathcal{A}, I \models \psi \wedge \phi$ iff $\mathcal{A}, I \models \psi$ and $\mathcal{A}, I \models \phi$;
- $\mathcal{A}, I \models \langle X \rangle \psi$, for $X \in \{A, B, E\}$, iff there is $J \in \mathbb{I}$ s.t. $I \, X_\mathbb{I} \, J$ and $\mathcal{A}, J \models \psi$;
- $\mathcal{A}, I \models \langle \overline{X} \rangle \psi$, for $\overline{X} \in \{\overline{A}, \overline{B}, \overline{E}\}$, iff there is $J \in \mathbb{I}$ s.t. $J \, X_\mathbb{I} \, I$ and $\mathcal{A}, J \models \psi$.

Kripke structures and abstract interval models. Finite state systems are usually modelled as finite Kripke structures. In [23], the authors define a mapping from Kripke structures to abstract interval models, that allows one to specify interval properties of computations by means of HS formulas.

Definition 2. *A* finite Kripke structure *is a tuple* $\mathcal{K} = (\mathcal{AP}, W, \delta, \mu, w_0)$*, where* $\mathcal{AP}$ *is a set of proposition letters,* W *is a finite set of states,* $\delta \subseteq W \times W$ *is a left-total relation between pairs of states,* $\mu : W \mapsto 2^{\mathcal{AP}}$ *is a total labelling function, and* $w_0 \in W$ *is the initial state.*

For all $w \in W$, $\mu(w)$ is the set of proposition letters that hold at w, while δ is the transition relation that describes the evolution of the system over time.

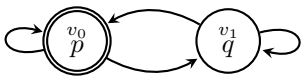


Fig. 1. The Kripke structure $\mathcal{K}_2$.

Fig. 1 depicts the finite Kripke structure $\mathcal{K}_2 = (\{p, q\}, \{v_0, v_1\}, \delta, \mu, v_0)$, where $\delta = \{(v_0, v_0), (v_0, v_1), (v_1, v_0), (v_1, v_1)\}$, $\mu(v_0) = \{p\}$, and $\mu(v_1) = \{q\}$. The initial state v_0 is identified by a double circle.

Definition 3. *A* track ρ *over a finite Kripke structure* $\mathcal{K} = (\mathcal{AP}, W, \delta, \mu, w_0)$ *is a finite sequence of states* $v_1 \cdots v_n$*, with* $n \geq 1$*, such that* $(v_i, v_{i+1}) \in \delta$ *for* $i = 1, \ldots, n-1$.

Let $\mathrm{Trk}_\mathcal{K}$ be the (infinite) set of all tracks over a finite Kripke structure $\mathcal{K}$. For any track $\rho = v_1 \cdots v_n \in \mathrm{Trk}_\mathcal{K}$, we define:

- $|\rho| = n$, $\mathrm{fst}(\rho) = v_1$, and $\mathrm{lst}(\rho) = v_n$;
- any index $i \in [1, |\rho|]$ is called a *ρ-position* and $\rho(i) = v_i$;
- $\mathrm{states}(\rho) = \{v_1, \cdots, v_n\} \subseteq W$;
- $\rho(i, j) = v_i \cdots v_j$, for $1 \leq i \leq j \leq |\rho|$, is the subtrack of ρ bounded by the ρ-positions i and j (we write ρ^i for $\rho(i, |\rho|)$, for $1 \leq i \leq |\rho|$);
- $\mathrm{Pref}(\rho) = \{\rho(1, i) \mid 1 \leq i \leq |\rho| - 1\}$ and $\mathrm{Suff}(\rho) = \{\rho(i, |\rho|) \mid 2 \leq i \leq |\rho|\}$ are the sets of all proper prefixes and suffixes of ρ, respectively.

Given $\rho, \rho' \in \mathrm{Trk}_\mathcal{K}$, we denote by $\rho \cdot \rho'$ the concatenation of the tracks ρ and ρ'. Moreover, if $\mathrm{lst}(\rho) = \mathrm{fst}(\rho')$, we denote by $\rho \star \rho'$ the track $\rho(1, |\rho| - 1) \cdot \rho'$. In particular, when $|\rho| = 1$, $\rho \star \rho' = \rho'$. In the following, when we write $\rho \star \rho'$, we implicitly assume that $\mathrm{lst}(\rho) = \mathrm{fst}(\rho')$. Finally, if $\mathrm{fst}(\rho) = w_0$ (the initial state of $\mathcal{K}$), ρ is called an *initial track*.

An abstract interval model (over $\mathrm{Trk}_{\mathcal{K}}$) can be naturally associated with a finite Kripke structure $\mathcal{K}$ by considering the set of intervals as the set of tracks of $\mathcal{K}$. Since $\mathcal{K}$ has loops (δ is left-total), the number of tracks in $\mathrm{Trk}_{\mathcal{K}}$, and thus the number of intervals, is infinite.

Definition 4. *The* abstract interval model induced by a finite Kripke structure $\mathcal{K} = (\mathcal{AP}, W, \delta, \mu, w_0)$ *is* $\mathcal{A}_{\mathcal{K}} = (\mathcal{AP}, \mathbb{I}, A_{\mathbb{I}}, B_{\mathbb{I}}, E_{\mathbb{I}}, \sigma)$, *where* $\mathbb{I} = \mathrm{Trk}_{\mathcal{K}}$, $A_{\mathbb{I}} = \{(\rho, \rho') \in \mathbb{I} \times \mathbb{I} \mid \mathrm{lst}(\rho) = \mathrm{fst}(\rho')\}$, $B_{\mathbb{I}} = \{(\rho, \rho') \in \mathbb{I} \times \mathbb{I} \mid \rho' \in \mathrm{Pref}(\rho)\}$, $E_{\mathbb{I}} = \{(\rho, \rho') \in \mathbb{I} \times \mathbb{I} \mid \rho' \in \mathrm{Suff}(\rho)\}$, *and* $\sigma : \mathbb{I} \mapsto 2^{\mathcal{AP}}$ *is such that* $\sigma(\rho) = \bigcap_{w \in \mathrm{states}(\rho)} \mu(w)$, *for all* $\rho \in \mathbb{I}$.

Relations $A_{\mathbb{I}}, B_{\mathbb{I}}$, and $E_{\mathbb{I}}$ are interpreted as the Allen's relations A, B, and E, respectively. Moreover, according to the definition of σ, $p \in \mathcal{AP}$ holds over $\rho = v_1 \cdots v_n$ iff it holds over all the states $v_1, \cdots, v_n$ of ρ. This conforms to the *homogeneity principle*, according to which a proposition letter holds over an interval if and only if it holds over all its subintervals.

Definition 5. *Let $\mathcal{K}$ be a finite Kripke structure and ψ be an HS formula; we say that a track $\rho \in \mathrm{Trk}_{\mathcal{K}}$ satisfies ψ, denoted as $\mathcal{K}, \rho \models \psi$, iff it holds that $\mathcal{A}_{\mathcal{K}}, \rho \models \psi$. Moreover, we say that $\mathcal{K}$ models ψ, denoted as $\mathcal{K} \models \psi$, iff for all* initial *tracks $\rho' \in \mathrm{Trk}_{\mathcal{K}}$ it holds that $\mathcal{K}, \rho' \models \psi$. The* model checking problem *for HS over finite Kripke structures is the problem of deciding whether $\mathcal{K} \models \psi$.*

We conclude with a simple example (a simplified version of the one given in [19]), showing that the fragments considered in this paper can express meaningful properties of state-transition systems.

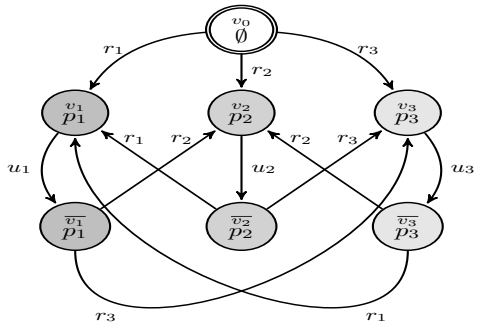


Fig. 2. The Kripke structure $\mathcal{K}_{Sched}$.

In Fig. 2, we provide an example of a finite Kripke structure $\mathcal{K}_{Sched}$ that models the behaviour of a scheduler serving three processes which are continuously requesting the use of a common resource. The initial state is v_0: no process is served in that state. In any other state v_i and $\overline{v}_i$, with $i \in \{1, 2, 3\}$, the i-th process is served (this is denoted by the fact that p_i holds in those states). For the sake of readability, edges are marked either by r_i, for $request(i)$, or by u_i, for $unlock(i)$. Edge labels do not have a semantic value, that is, they are neither part of the structure definition, nor proposition letters; they are simply used to ease reference to edges. Process i is served in state v_i, then, after "some time", a transition u_i from v_i to $\overline{v}_i$ is taken; subsequently, process i cannot be served again immediately, as v_i is not directly reachable from $\overline{v}_i$ (the scheduler cannot serve the same process twice in two successive rounds). A transition r_j, with $j \neq i$, from $\overline{v}_i$ to v_j is then taken and process j is served. This structure can easily be generalised to a higher number of processes.

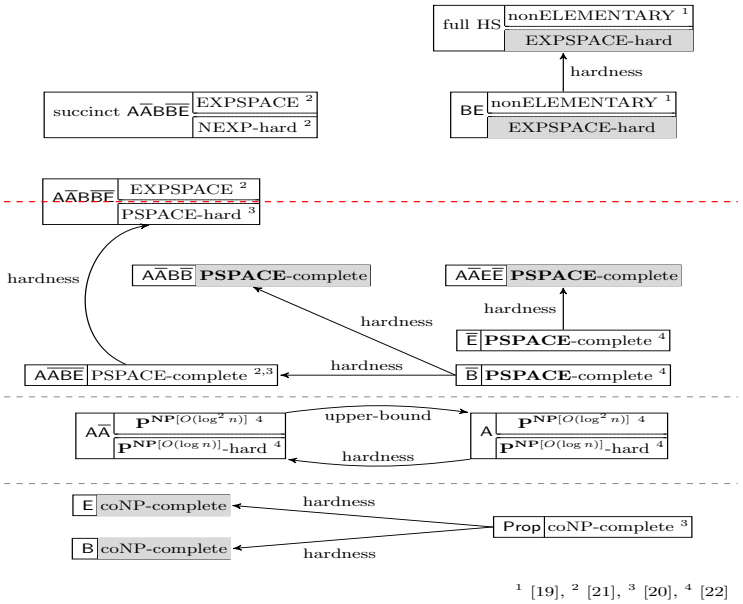


Fig. 3. Complexity of the model checking problem for HS fragments: known results are depicted in white boxes, new ones in gray boxes.

We show how some meaningful properties to be checked over $\mathcal{K}_{Sched}$ can be expressed in HS, and, in particular, with formulas of $\mathsf{A\overline{A}E\overline{E}}$. In all formulas, we force the validity of the considered property over all legal computation sub-intervals by using modality $[E]$ (all computation sub-intervals are suffixes of at least one initial track). The truth of the next statements can be easily checked:

- $\mathcal{K}_{Sched} \models [E]\big(\langle \mathrm{E}\rangle^3 \top \rightarrow (\chi(p_1,p_2) \vee \chi(p_1,p_3) \vee \chi(p_2,p_3))\big)$, where $\chi(p,q) := \langle \mathrm{E}\rangle \langle \overline{\mathrm{A}}\rangle p \wedge \langle \mathrm{E}\rangle \langle \overline{\mathrm{A}}\rangle q$;
- $\mathcal{K}_{Sched} \not\models [E](\langle \mathrm{E}\rangle^{10} \top \rightarrow \langle \mathrm{E}\rangle \langle \overline{\mathrm{A}}\rangle p_3)$;
- $\mathcal{K}_{Sched} \not\models [E](\langle \mathrm{E}\rangle^{5} \rightarrow (\langle \mathrm{E}\rangle \langle \overline{\mathrm{A}}\rangle p_1 \wedge \langle \mathrm{E}\rangle \langle \overline{\mathrm{A}}\rangle p_2 \wedge \langle \mathrm{E}\rangle \langle \overline{\mathrm{A}}\rangle p_3))$.

The first formula states that in any suffix of length at least 4 of an initial track, at least 2 proposition letters are witnessed. $\mathcal{K}_{Sched}$ satisfies the formula since a process cannot be executed twice in a row. The second formula states that in any suffix of length at least 11 of an initial track, process 3 is executed at least once in some internal states (*non starvation*). $\mathcal{K}_{Sched}$ does not satisfy the formula since the scheduler can avoid executing a process ad libitum. The third formula states that in any suffix of length at least 6 of an initial track, p_1, p_2, p_3 are all witnessed. The only way to satisfy this property is to constrain the scheduler to execute the three processes in a strictly periodic manner (*strict alternation*), i.e., $p_i p_j p_k p_i p_j p_k p_i p_j p_k \cdots$, $i,j,k \in \{1,2,3\}, i \neq j \neq k \neq i$, but this is not the case.

The general picture. We now describe known and new complexity results about the model checking problem for HS fragments (see Fig. 3 for a visual account).

In [19, 23], the authors show that, given a finite Kripke structure $\mathcal{K}$ and a bound k on the structural complexity of HS formulas, that is, on the nesting depth of $\langle \mathrm{E} \rangle$ and $\langle \mathrm{B} \rangle$ modalities, it is possible to obtain a *finite* representation for $\mathcal{A}_{\mathcal{K}}$, which is equivalent to $\mathcal{A}_{\mathcal{K}}$ with respect to satisfiability of HS formulas with structural complexity less than or equal to k. Then, by exploiting such a representation, they prove that the model checking problem for (full) HS is decidable, providing an algorithm with non-elementary complexity. Moreover, they show that the problem for the fragment $\mathsf{A\overline{A}BE}$, and thus for full HS, is **PSPACE**-hard (**EXPSPACE**-hard if a suitable succinct encoding of formulas is exploited). In [21], the authors study the HS fragments $\mathsf{A\overline{A}B\overline{B}E}$ and $\mathsf{A\overline{A}E\overline{B}E}$, devising for each of them an **EXPSPACE** model checking algorithm which exploits the possibility of finding, for each track of the Kripke structure, a satisfiability-preserving track of bounded length, called *track representative*. In this way, the algorithm needs to check only tracks having a bounded maximum length. Later [20], they prove that the problem for $\mathsf{A\overline{A}B\overline{B}E}$ and $\mathsf{A\overline{A}E\overline{B}E}$ is **PSPACE**-hard (if a suitable succinct encoding of formulas is exploited, the algorithm remains in **EXPSPACE**, but a **NEXPTIME** lower bound can be given [21]). Finally, they show that formulas satisfying a constant bound on the nesting depth of $\langle \mathrm{B} \rangle$ (resp., $\langle \mathrm{E} \rangle$) can be checked in polynomial working space [21].

In [20, 22] the authors identify some well-behaved HS fragments, namely, $\mathsf{A\overline{A}B\overline{E}}$, $\mathsf{\overline{B}}$, $\mathsf{\overline{E}}$, $\mathsf{A\overline{A}}$, A, and $\mathsf{\overline{A}}$, which are still expressive enough to capture meaningful interval properties of state-transition systems and whose model checking problem has a computational complexity markedly lower than that of full HS. In particular, they prove that the problem is **PSPACE**-complete for the first three fragments, and in between $\mathbf{P}^{\mathbf{NP}[O(\log n)]}$ and $\mathbf{P}^{\mathbf{NP}[O(\log^2 n)]}$ [10, 29] for the last three. In all cases, the complexity of the problem turns out be comparable to or lower than that of LTL, which is known to be **PSPACE**-complete [30].

In this paper, we first strengthen the lower bound to the complexity of the model checking problem for full HS by proving **EXPSPACE**-hardness of the fragment BE. Then, we study two more well-behaved fragments, namely, $\mathsf{A\overline{A}B\overline{B}}$ and $\mathsf{A\overline{A}E\overline{E}}$, and we prove that their model checking problem is **PSPACE**-complete (the previously known upper bound was **EXPSPACE** [21]). This is somehow surprising, as their expressive power seems to be really higher than that of the fragments analyzed in [20, 22], but their complexity turns out to be the same. Finally, we prove that B and E are in **co-NP**, and thus **co-NP**-complete, as the purely propositional fragment of HS, Prop, is **co-NP**-complete [20].

It is worth pointing out that, in order to determine the complexity of $\mathsf{A\overline{A}B\overline{B}}$ and $\mathsf{A\overline{A}E\overline{E}}$, we exploit the structure of the specific input formula, rather than considering generically the nesting depth of $\langle \mathrm{B} \rangle$ or $\langle \mathrm{E} \rangle$ modalities (as we did in [21]). In [21] a *track representative* is a track of *exponential length*, which is satisfiability equivalent—with respect to *all* $\mathsf{A\overline{A}B\overline{B}E}$ (resp., $\mathsf{A\overline{A}E\overline{B}E}$) formulas with nesting depth of $\langle \mathrm{B} \rangle$ (resp., $\langle \mathrm{E} \rangle$) modality equal to or less than some k—to all the (possibly infinitely many) represented tracks. Here, we weaken such a strong constraint by requiring satisfiability equivalence only with respect to the *specific* formula under consideration, which allows us to restrict our attention to

tracks of *polynomially-bounded length*, that is, we prove that if a track ρ fulfils a formula ψ of $\mathsf{A\overline{A}B\overline{B}}$ (resp., $\mathsf{A\overline{A}E\overline{E}}$), then there is a *polynomial-length* track ρ' satisfying ψ as well (such a track depends on ψ).

3 EXPSPACE-hardness of BE

In this section, we prove that the model checking problem for formulas of the HS fragment BE is **EXPSPACE**-hard. This lower-bound immediately propagates to the problem for full HS formulas.

Theorem 1. *The model-checking problem for* BE *formulas over finite Kripke structures is* **EXPSPACE***-hard (under polynomial-time reductions).*

Proof. The claim is proved by a polynomial-time reduction from a domino-tiling problem for grids with rows of single exponential length [12]. An instance $\mathcal{I}$ of such problem is a tuple $\mathcal{I} = (C, \Delta, n, d_{init}, d_{final})$, where C is a finite set of colors, $\Delta \subseteq C^4$ is a set of tuples $(c_{down}, c_{left}, c_{up}, c_{right})$ of four colors, called *domino-types*, $n > 0$ is a natural number encoded in *unary*, and $d_{init}, d_{final} \in \Delta$ are domino-types. A *tiling of* $\mathcal{I}$ is a mapping $f : [0, k] \times [0, 2^n - 1] \to \Delta$, for some $k \geq 0$, satisfying the following constraints:

- two adjacent cells in a row have the same color on the shared edge: for all $(i, j) \in [0, k] \times [0, 2^n - 2]$, $[f(i, j)]_{right} = [f(i, j + 1)]_{left}$;
- two adjacent cells in a column have the same color on the shared edge: for all $(i, j) \in [0, k - 1] \times [0, 2^n - 1]$, $[f(i, j)]_{up} = [f(i + 1, j)]_{down}$;
- $f(0, 0) = d_{init}$ (*initialization*) and $f(k, 2^n - 1) = d_{final}$ (*acceptance*).

It is well-known that checking the existence (resp., the non-existence) of a tiling for $\mathcal{I}$ is **EXPSPACE**-complete [12]. We now show how to build in polynomial time a Kripke structure $\mathcal{K}_\mathcal{I}$ and a BE formula $\varphi_\mathcal{I}$ such that there exists an initial track of $\mathcal{K}_\mathcal{I}$ satisfying $\varphi_\mathcal{I}$ if and only if there exists a tiling of $\mathcal{I}$. Hence $\mathcal{K}_\mathcal{I} \models \neg\varphi_\mathcal{I}$ iff there does not exist a tiling of $\mathcal{I}$, and Theorem 1 follows.

We use the following set $\mathcal{AP}$ of proposition letters to encode tilings of $\mathcal{I}$: $\mathcal{AP} = \Delta \cup \{\$\} \cup \{0, 1\}$. Proposition letters in $\{0, 1\}$ are used to encode the value of an n-bits counter numbering the cells of one row of a tiling. In particular, a cell with content $d \in \Delta$ and column number $j \in [0, 2^n - 1]$ is encoded by the word of length $n + 1$ over $\mathcal{AP}$ given by $d\, b_1 \dots b_n$, where $b_1 \dots b_n$ is the binary encoding of the column number j (b_n is the most significant bit). A row is encoded by the word listing the encodings of cells from left to right, and a tiling f with $k + 1$ rows is encoded by the finite word $r_0 \$ r_1 \dots \$ r_k$, where r_i is the encoding of the i-th row of f for all $i \in [0, k]$.

The Kripke structure $\mathcal{K}_\mathcal{I}$ is defined as $\mathcal{K}_\mathcal{I} = (\mathcal{AP}, \mathcal{AP}, \mathcal{AP} \times \mathcal{AP}, \mu, d_{init})$, where $\mu(p) = \{p\}$, for any $p \in \mathcal{AP}$. Thus, the initial tracks of $\mathcal{K}_\mathcal{I}$ correspond to the finite words over $\mathcal{AP}$ which start with the initial domino type d_{init}.

In order to build the BE formula $\varphi_\mathcal{I}$, we use some auxiliary formulas, namely, $length_i$, $beg(p)$, $end(p)$, ϕ_{cell}, and $\theta_j(b, b')$ where $i \in [1, 2n + 2]$, $j \in [2, n + 1]$, $p \in \mathcal{AP}$, and $b, b' \in \{0, 1\}$. The formula $length_i$ has size linear in i and characterizes

the tracks of length i. It can be expressed as follows:

$$\mathit{length}_i := (\underbrace{\langle \mathrm{B} \rangle \dots \langle \mathrm{B} \rangle}_{i-1} \top) \ \wedge \ (\underbrace{[B] \dots [B]}_{i} \bot).$$

The formula $\mathit{beg}(p)$ (resp., $\mathit{end}(p)$) captures the tracks of $\mathcal{K}$ which start (resp., end) in state p:

$$\mathit{beg}(p) := (p \wedge \mathit{length}_1) \vee \langle \mathrm{B} \rangle (p \wedge \mathit{length}_1), \quad \mathit{end}(p) := (p \wedge \mathit{length}_1) \vee \langle \mathrm{E} \rangle (p \wedge \mathit{length}_1).$$

The formula ϕ_{cell} captures the tracks of $\mathcal{K}_{\mathcal{I}}$ which encode cells:

$$\phi_{\mathit{cell}} := \mathit{length}_{n+1} \wedge \Big(\bigvee_{d \in \Delta} \mathit{beg}(d) \Big) \wedge [E](\mathit{beg}(0) \vee \mathit{beg}(1)).$$

Finally, for all $j \in [2, n+1]$ and $b, b' \in \{0, 1\}$, the formula $\theta_j(b, b')$ is defined as $\theta_j(b, b') := \langle \mathrm{B} \rangle (\mathit{length}_j \wedge \mathit{end}(b)) \wedge \langle \mathrm{E} \rangle (\mathit{length}_{n-j+2} \wedge \mathit{beg}(b'))$. The formula $\theta_j(b, b')$ is satisfied by a track ρ if $|\rho| \geq j+1$, $|\rho| \geq n-j+3$, $\rho(j) = b$, and $\rho(|\rho| - n + j - 1) = b'$. In particular, for a track ρ starting with a cell c and ending with a cell c', $\theta_j(b, b')$ is satisfied by ρ if the jth bit of c is b and the jth bit of c' is b'.

Additionally, we use the derived operator $\langle \mathrm{G} \rangle$ and its dual $[G]$, which allow us to select arbitrary subtracks of the given track, including the track itself:

$$\langle \mathrm{G} \rangle \, \psi := \psi \vee \langle \mathrm{B} \rangle \, \psi \vee \langle \mathrm{E} \rangle \, \psi \vee \langle \mathrm{B} \rangle \, \langle \mathrm{E} \rangle \, \psi.$$

Then, the formula $\varphi_{\mathcal{I}}$ is defined as $\varphi_{\mathcal{I}} := \varphi_b \wedge \varphi_{req} \wedge \varphi_{inc} \wedge \varphi_{rr} \wedge \varphi_{rc}$.

φ_b checks that the given track starts with a cell with content d_{init} and column number 0, and ends with a cell with content d_{final} and column number $2^n - 1$:

$$\varphi_b := \langle \mathrm{B} \rangle \, \phi_{\mathit{cell}} \wedge \mathit{beg}(d_{\mathit{init}}) \wedge \langle \mathrm{E} \rangle (\phi_{\mathit{cell}} \wedge \mathit{beg}(d_{\mathit{final}})) \wedge \bigwedge_{j=2}^{n+1} \theta_j(0, 1).$$

The conjunct φ_{req} ensures the following two requirements: (i) each occurrence of \$ in the given track is followed by a cell with column number 0 and (ii) each cell c in the given track is followed either by another cell, or by the separator \$, and in the latter case c has column number $2^n - 1$. The first requirement is encoded by the formula: $[G]\big((\mathit{length}_{n+2} \wedge \mathit{beg}(\$)) \rightarrow \langle \mathrm{E} \rangle (\phi_{\mathit{cell}} \wedge [E] \mathit{beg}(0))\big)$; the second one by the formula:

$$[G]\Big\{ (\mathit{length}_{n+2} \wedge \bigvee_{d \in \Delta} \mathit{beg}(d)) \rightarrow \Big(\langle \mathrm{B} \rangle \, \phi_{\mathit{cell}} \wedge (\mathit{end}(\$) \vee \bigvee_{d \in \Delta} \mathit{end}(d)) \wedge (\mathit{end}(\$) \rightarrow [E](\mathit{beg}(\$) \vee \mathit{beg}(1))) \Big) \Big\}.$$

The conjunct φ_{inc} checks that adjacent cells along the given track have consecutive columns numbers:

$$\varphi_{inc} = [G]\Big(\phi_{\mathit{two\_cells}} \rightarrow \bigvee_{j=2}^{n+1} \big[\theta_j(0, 1) \wedge \bigwedge_{h=2}^{j-1} \theta_h(1, 0) \wedge \bigwedge_{h=j+1}^{n+1} \bigvee_{b \in \{0,1\}} \theta_h(b, b) \big] \Big),$$

where ϕ_{two_cells} is given by $length_{2n+2} \wedge \langle \mathrm{B} \rangle\, \phi_{cell} \wedge \langle \mathrm{E} \rangle\, \phi_{cell}$. Note that φ_{req} and φ_{inc} ensure that the column numbers are correctly encoded.

The conjunct φ_{rr} checks that adjacent cells in a row have the same color on the shared edge:

$$\varphi_{rr} = [G]\Big(\phi_{two\_cells} \rightarrow \bigvee_{(d,d') \in \Delta \times \Delta \mid d_{right} = d'_{left}} (beg(d) \wedge \langle \mathrm{E} \rangle (length_{n+1} \wedge beg(d')))\Big).$$

Finally, the conjunct φ_{rc} checks that adjacent cells in a column have the same color on the shared edge. For this, it suffices to require that for each subtrack of the given one containing exactly one occurrence of \$, starting with a cell c, and ending with a cell c', if c and c' have the same column number, then $d_{up} = d'_{down}$, where d (resp., d') is the content of c (resp., c'). Thus, formula φ_{rc} is defined as follows, where we use the formulas $\theta_j(b,b)$, with $j \in [2, n+1]$ and $b \in \{0,1\}$, for expressing that c and c' have the same column number:

$$\varphi_{rc} = [G]\Big\{ \Big(\phi_{one}(\$) \ \wedge\ \langle \mathrm{B} \rangle\, \phi_{cell} \ \wedge\ \langle \mathrm{E} \rangle\, \phi_{cell} \ \wedge \bigwedge_{j=2}^{n+1} \bigvee_{b \in \{0,1\}} \theta_j(b,b)\ \Big) \rightarrow \bigvee_{(d,d') \in \Delta \times \Delta \mid d_{up} = d'_{down}} (beg(d) \wedge \langle \mathrm{E} \rangle (length_{n+1} \wedge beg(d')))\ \Big\},$$

where $\phi_{one}(\$)$ is defined as $(\langle \mathrm{B} \rangle\, end(\$)) \ \wedge\ \neg(\langle \mathrm{B} \rangle (end(\$) \wedge \langle \mathrm{B} \rangle\, end(\$)))$.

Note that $\varphi_{\mathcal{I}}$ has size polynomial in the size of $\mathcal{I}$. By construction, a track ρ of $\mathcal{K}_{\mathcal{I}}$ satisfies $\varphi_{\mathcal{I}}$ if and only if ρ encodes a tiling. Since the initial tracks of $\mathcal{K}_{\mathcal{I}}$ are the finite words over $\mathcal{AP}$ starting with d_{init}, it follows that there exists a tiling of $\mathcal{I}$ if and only if there exists an initial track of $\mathcal{K}_{\mathcal{I}}$ which satisfies $\varphi_{\mathcal{I}}$. Hence, the result follows, which concludes the proof. □

4 The fragments $\mathsf{A\overline{A}E\overline{E}}$ and $\mathsf{A\overline{A}B\overline{B}}$: polynomial-size model-track property

In this section, we show that the model checking problem for the fragments $\mathsf{A\overline{A}E\overline{E}}$ and $\mathsf{A\overline{A}B\overline{B}}$ is in **PSPACE** by proving that a *polynomial size model-track property* holds, that is, we show that if a track ρ of a Kripke structure $\mathcal{K}$ satisfies a given formula φ of the fragments $\mathsf{A\overline{A}E\overline{E}}$ or $\mathsf{A\overline{A}B\overline{B}}$, then there exists also a track π, whose length is polynomial in the sizes of φ and $\mathcal{K}$, starting from and leading to the same states as ρ, that satisfies φ. Moreover, we show that the problem is in **co-NP** for the smaller fragments B and E. We conclude the section by providing two model checking procedures, one for $\mathsf{A\overline{A}E\overline{E}}$ formulas and one for E formulas.

In the following, we focus on the fragment $\mathsf{A\overline{A}E\overline{E}}$ and the smaller fragment E, being the cases of the fragments $\mathsf{A\overline{A}B\overline{B}}$ and B completely symmetric.

Let $\mathcal{K} = (\mathcal{AP}, W, \delta, \mu, w_0)$ be a Kripke structure. We start by introducing the notions of *induced track* and *well-formed track*, which will be exploited to prove the polynomial size model-track property.

Definition 6. *Let $\rho \in \mathrm{Trk}_{\mathcal{K}}$ be a track of length n. A* track induced by ρ *is a track $\pi \in \mathrm{Trk}_{\mathcal{K}}$ such that there exists an increasing sequence of ρ-positions $i_1 < \ldots < i_k$, with $i_1 = 1$, $i_k = n$, and $\pi = \rho(i_1) \cdots \rho(i_k)$. Moreover, we say that the π-position j and the ρ-position i_j are corresponding.*
The induced track π is well-formed with respect to ρ *if, for all π-positions j, with corresponding ρ-positions i_j, and all proposition letters $p \in \mathcal{AP}$, it holds that $\mathcal{K}, \pi^j \models p \iff \mathcal{K}, \rho^{i_j} \models p$.*

Note that if π is induced by ρ, then $\mathrm{fst}(\pi) = \mathrm{fst}(\rho)$, $\mathrm{lst}(\pi) = \mathrm{lst}(\rho)$, and $|\pi| \leq |\rho|$ (in particular, $|\pi| = |\rho|$ iff $\pi = \rho$). Intuitively, a track induced by ρ is obtained by contracting ρ, namely, by concatenating some subtracks of ρ, provided that the resulting sequence is a track of $\mathcal{K}$ as well. Well-formedness implies that the suffix of π starting from position j and the suffix of ρ starting from the corresponding position i_j agree over all the proposition letters in $\mathcal{AP}$, i.e., they have the same satisfiability pattern of proposition letters. In particular, $\mathcal{K}, \pi \models p$ iff $\mathcal{K}, \rho \models p$, for all $p \in \mathcal{AP}$. It can be easily seen that *the well-formedness relation is transitive.*

The following proposition shows how it is possible to contract a track, preserving the same satisfiability of proposition letters with respect to suffixes. Such a criterion represents a "basic step" in a contraction process which will allow us to prove the polynomial size model-track property.

Proposition 1. *For any track ρ of $\mathcal{K} = (\mathcal{AP}, W, \delta, \mu, w_0)$, there exists a track π of $\mathcal{K}$, which is well-formed with respect to ρ, such that $|\pi| \leq |W| \cdot (|\mathcal{AP}| + 1)$.*

Proof. Let $\rho \in \mathrm{Trk}_{\mathcal{K}}$ be a track of length n. If $n \leq |W| \cdot (|\mathcal{AP}| + 1)$, the thesis trivially holds. Let us assume $n > |W| \cdot (|\mathcal{AP}| + 1)$. We show that there exists a track of $\mathcal{K}$ which is well-formed with respect to ρ and whose length is smaller than n. Since $n > |W| \cdot (|\mathcal{AP}| + 1)$, there is some state $w \in W$ occurring in ρ at least $|\mathcal{AP}| + 2$ times. Assume that for all ρ-positions i and j, with $j > i$, if $\rho(i) = \rho(j) = w$, then there exists some $p \in \mathcal{AP}$ such that $\mathcal{K}, \rho^j \models p$ and $\mathcal{K}, \rho^i \not\models p$. This assumption leads to a contradiction, as the suffixes of ρ may feature at most $|\mathcal{AP}|+1$ distinct satisfiability patterns of proposition letters (due to the homogeneity principle in Definition 4), while there are at least $|\mathcal{AP}| + 2$ occurrences of w. As a consequence, there are two ρ-positions i and j, with $j > i$, such that $\rho(i) = \rho(j) = w$ and, for all $p \in \mathcal{AP}$, $\mathcal{K}, \rho^j \models p$ iff $\mathcal{K}, \rho^i \models p$. It is easy to see that $\pi = \rho(1, i) \star \rho(j, n) \in \mathrm{Trk}_{\mathcal{K}}$ is well-formed with respect to ρ and $|\pi| < n$. Now, if $|\pi| \leq |W| \cdot (|\mathcal{AP}| + 1)$, the thesis is proved; otherwise, the same basic step can be iterated a finite number of times, and the thesis follows by transitivity of the well-formedness relation. □

The next definition introduces some distinguished positions in a track. The intuition is that—as we will see in the proof of Theorem 2—if we perform a contraction (as we did in the proof of Proposition 1) between a pair of such positions, we get an equivalent track with respect to satisfiability of the considered $\mathsf{A\overline{A}E\overline{E}}$ formula. In the following, we restrict ourselves to formulas in *negation normal form* (NNF), namely, formulas where negation is applied only to proposition letters. By using De Morgan's laws and the dual modalities $[E]$, $[\overline{E}]$, $[A]$, and

$[\overline{A}]$ of $\langle \mathrm{E} \rangle$, $\langle \overline{\mathrm{E}} \rangle$, $\langle \mathrm{A} \rangle$, and $\langle \overline{\mathrm{A}} \rangle$, we can trivially convert in linear time a formula into an equivalent one in NNF, of at most double length.

Definition 7 (Witness positions). *Let ρ be a track of $\mathcal{K}$ and φ be a formula of $\mathsf{A}\overline{\mathsf{A}}\mathsf{E}\overline{\mathsf{E}}$. Let us denote by $E(\varphi, \rho)$ the set of subformulas $\langle \mathrm{E} \rangle \psi$ of φ such that $\mathcal{K}, \rho \models \langle \mathrm{E} \rangle \psi$. The* set $Wt(\varphi, \rho)$ of witness positions of ρ for φ *is the* minimal *set of ρ-positions satisfying the following constraint: for each $\langle \mathrm{E} \rangle \psi \in E(\varphi, \rho)$, the greatest ρ-position $i > 1$ such that $\mathcal{K}, \rho^i \models \psi$ belongs to $Wt(\varphi, \rho)$*[6]*.*

It is immediate to see that the cardinalities of $E(\varphi, \rho)$ and of $Wt(\varphi, \rho)$ are at most $|\varphi| - 1$. We are now ready to prove the polynomial-size model-track property.

Theorem 2 (Polynomial-size model-track property). *Let ρ and σ be two tracks of $\mathcal{K} = (\mathcal{AP}, W, \delta, \mu, w_0)$ and φ be an $\mathsf{A}\overline{\mathsf{A}}\mathsf{E}\overline{\mathsf{E}}$ formula in NNF such that $\mathcal{K}, \rho \star \sigma \models \varphi$. Then, there exists a track π, induced by ρ, such that $\mathcal{K}, \pi \star \sigma \models \varphi$, and $|\pi| \leq |W| \cdot (|\varphi| + 1)^2$.*

Notice that the theorem holds in particular if $|\sigma| = 1$, and thus $\rho \star \sigma = \rho$ and $\pi \star \sigma = \pi$. In such a case, if $\mathcal{K}, \rho \models \varphi$, then $\mathcal{K}, \pi \models \varphi$, where π is induced by ρ and $|\pi| \leq |W| \cdot (|\varphi| + 1)^2$. The more general statement of Theorem 2 is needed for technical reasons in the soundness/completeness proof of the next algorithms.

Proof. W.l.o.g., we restrict ourselves to the proposition letters occurring in φ. Thus, $|\mathcal{AP}| \leq |\varphi|$. Let $Wt(\varphi, \rho \star \sigma)$ be the set of witness positions of $\rho \star \sigma$ for φ. Let $\{i_1, \ldots, i_k\}$ be the ordering of $Wt(\varphi, \rho \star \sigma)$ such that $i_1 < \ldots < i_k$. Let $i_0 = 1$ and $i_{k+1} = |\rho \star \sigma|$. Hence, $1 = i_0 < i_1 < \ldots < i_k \leq i_{k+1} = |\rho \star \sigma|$.

If the length of ρ is at most $|W| \cdot (|\varphi| + 1)^2$, the thesis trivially holds. Let us assume that $|\rho| > |W| \cdot (|\varphi| + 1)^2$. We show that there exists a track π induced by ρ, with $|\pi| < |\rho|$, such that $\mathcal{K}, \pi \star \sigma \models \varphi$.

W.l.o.g., we can assume that $i_0 < i_1 < \ldots < i_j$, for some $j \geq 0$, are ρ-positions (while $i_{j+1} < \ldots < i_{k+1}$ are $(\rho \star \sigma)$-positions not in ρ). We claim that either (i) there exists $t \in [0, j-1]$ such that $i_{t+1} - i_t > |W| \cdot (|\varphi| + 1)$ or (ii) $|\rho(i_j, |\rho|)| > |W| \cdot (|\varphi| + 1)$. By way of contradiction, suppose that neither (i) nor (ii) holds. We need to distinguish two cases. If $\rho \star \sigma = \rho$, then $|\rho| = (i_{k+1} - i_0) + 1 \leq (k+1) \cdot |W| \cdot (|\varphi| + 1) + 1$; otherwise ($|\rho| < |\rho \star \sigma|$), $|\rho| = (i_j - i_0) + |\rho(i_j, |\rho|)| \leq j \cdot |W| \cdot (|\varphi| + 1) + |W| \cdot (|\varphi| + 1) \leq (k+1) \cdot |W| \cdot (|\varphi| + 1)$. The contradiction follows since $(k+1) \cdot |W| \cdot (|\varphi| + 1) + 1 \leq |\varphi| \cdot |W| \cdot (|\varphi| + 1) + 1 \leq |W| \cdot (|\varphi| + 1)^2$.

Let us define $(\alpha, \beta) = (i_t, i_{t+1})$ in case (i), and $(\alpha, \beta) = (i_j, |\rho|)$ in case (ii). Moreover let $\rho' = \rho(\alpha, \beta)$. In both cases, we have $|\rho'| > |W| \cdot (|\varphi| + 1) \geq |W| \cdot (|\mathcal{AP}| + 1)$, being $|\mathcal{AP}| \leq |\varphi|$. By Proposition 1, there exists a track π' of $\mathcal{K}$, well-formed with respect to ρ', such that $|\pi'| \leq |W| \cdot (|\mathcal{AP}| + 1) < |\rho'|$. Let π be the track induced by ρ obtained by replacing the subtrack ρ' of ρ with π'. Since $|\pi| < |\rho|$, it remains to prove that $\mathcal{K}, \pi \star \sigma \models \varphi$.

Let us denote $\pi \star \sigma$ by $\overline{\pi}$ and $\rho \star \sigma$ by $\overline{\rho}$. Moreover, let $H : [1, |\overline{\pi}|] \to [1, |\overline{\rho}|]$ be the function mapping positions of $\overline{\pi}$ into positions of $\overline{\rho}$ in this way: positions "outside" π' (i.e., outside the interval $[\alpha, \alpha + |\pi'| - 1]$) are mapped into their

[6] Note that such a ρ-position exists by definition of $E(\varphi, \rho)$.

original position in $\overline{\rho}$; positions "inside" π' (i.e., in $[\alpha, \alpha + |\pi'| - 1]$) are mapped to the corresponding position in ρ' (exploiting well-formedness of π' w.r. to ρ').

$$H(m) = \begin{cases} m & \text{if} \quad m < \alpha \\ \alpha + \ell_{m-\alpha+1} - 1 & \text{if} \quad \alpha \leq m < \alpha + |\pi'| \\ m + (|\rho'| - |\pi'|) & \text{if} \quad m \geq \alpha + |\pi'| \end{cases} \tag{1}$$

where ℓ_m is the ρ'-position corresponding to the π'-position m. It is easy to check that H satisfies the following properties:

1. H is strictly monotonic, i.e., for all $j, j' \in [1, |\overline{\pi}|]$, $j < j'$ iff $H(j) < H(j')$;
2. for all $j \in [1, |\overline{\pi}|]$, $\overline{\pi}(j) = \overline{\rho}(H(j))$;
3. $H(1) = 1$ and $H(|\overline{\pi}|) = |\overline{\rho}|$;
4. $Wt(\varphi, \overline{\rho}) \subseteq \{H(j) \mid j \in [1, |\overline{\pi}|]\}$;
5. for each $j \in [1, |\overline{\pi}|]$ and $p \in \mathcal{AP}$, $\mathcal{K}, \overline{\pi}^j \models p$ iff $\mathcal{K}, \overline{\rho}^{H(j)} \models p$.

The fact that $\mathcal{K}, \overline{\pi} \models \varphi$ is an immediate consequence of the following claim, considering that $H(1) = 1$, $\mathcal{K}, \overline{\rho} \models \varphi$, $\overline{\rho}^1 = \overline{\rho}$, and $\overline{\pi}^1 = \overline{\pi}$.

Claim. For all $j \in [1, |\overline{\pi}|]$, all subformulas ψ of φ, and all $u \in \mathrm{Trk}_{\mathcal{K}}$, it holds that if $\mathcal{K}, u \star \overline{\rho}^{H(j)} \models \psi$, then $\mathcal{K}, u \star \overline{\pi}^j \models \psi$.

Proof. Assume that $\mathcal{K}, u \star \overline{\rho}^{H(j)} \models \psi$. Note that $u \star \overline{\rho}^{H(j)}$ is defined iff $u \star \overline{\pi}^j$ is defined. We prove by induction on the structure of φ that $\mathcal{K}, u \star \overline{\pi}^j \models \psi$. Since φ is in NNF, only the following cases occur:

- $\psi = p$ or $\psi = \neg p$ for some $p \in \mathcal{AP}$. By Property 5 of H, $\mathcal{K}, \overline{\pi}^j \models p$ iff $\mathcal{K}, \overline{\rho}^{H(j)} \models p$. Hence, $\mathcal{K}, u \star \overline{\pi}^j \models p$ iff $\mathcal{K}, u \star \overline{\rho}^{H(j)} \models p$, and the result holds.
- $\psi = \theta_1 \wedge \theta_2$ or $\psi = \theta_1 \vee \theta_2$, for some $\mathsf{A\overline{A}E\overline{E}}$ formulas θ_1 and θ_2: the result directly follows from the inductive hypothesis.
- $\psi = [E]\theta$. We need to show that for each proper suffix η of $u \star \overline{\pi}^j$, $\mathcal{K}, \eta \models \theta$. We distinguish two cases:
  - η is *not* a proper suffix of $\overline{\pi}^j$. Hence, η is of the form $u^h \star \overline{\pi}^j$ for some $h \in [2, |u|]$. Since $\mathcal{K}, u \star \overline{\rho}^{H(j)} \models [E]\theta$, then $\mathcal{K}, u^h \star \overline{\rho}^{H(j)} \models \theta$. By induction, $\mathcal{K}, u^h \star \overline{\pi}^j \models \theta$.
  - η is a proper suffix of $\overline{\pi}^j$. Hence, $\eta = \overline{\pi}^h$ for some $h \in [j+1, |\overline{\pi}|]$. By Property 1 of H, $H(h) > H(j)$, and since $\mathcal{K}, u \star \overline{\rho}^{H(j)} \models \psi$, we have that $\mathcal{K}, \overline{\rho}^{H(h)} \models \theta$. By induction, $\mathcal{K}, \overline{\pi}^h \models \theta$.

  Therefore, $\mathcal{K}, u \star \overline{\pi}^j \models [E]\theta$.
- $\psi = \langle \mathrm{E} \rangle\, \theta$. We need to show that there exists a proper suffix of $u \star \overline{\pi}^j$ satisfying θ. Since $\mathcal{K}, u \star \overline{\rho}^{H(j)} \models \psi$, there exists a proper suffix η' of $u \star \overline{\rho}^{H(j)}$ such that $\mathcal{K}, \eta' \models \theta$. We distinguish two cases:
  - η' is *not* a proper suffix of $\overline{\rho}^{H(j)}$. Hence, η' is of the form $u^h \star \overline{\rho}^{H(j)}$ for some $h \in [2, |u|]$. By induction, $\mathcal{K}, u^h \star \overline{\pi}^j \models \theta$, and $\mathcal{K}, u \star \overline{\pi}^j \models \langle \mathrm{E} \rangle\, \theta$.
  - η' is a proper suffix of $\overline{\rho}^{H(j)}$. Hence, $\eta' = \overline{\rho}^i$ for some $i \in [H(j)+1, |\overline{\rho}|]$, and $\mathcal{K}, \overline{\rho}^i \models \theta$. Let i' be the greatest position of $\overline{\rho}$ such that $\mathcal{K}, \overline{\rho}^{i'} \models \theta$. Hence $i' \geq i$ and, by Definition 7, $i' \in Wt(\varphi, \overline{\rho})$. By Property 4 of H, $i' = H(h)$ for some $\overline{\pi}$-position h. Since $H(h) > H(j)$, it holds that $h > j$ (Property 1). By induction, $\mathcal{K}, \overline{\pi}^h \models \theta$, and $\mathcal{K}, u \star \overline{\pi}^j \models \langle \mathrm{E} \rangle\, \theta$.

– $\psi = [\overline{E}]\theta$ or $\psi = \langle\overline{\mathrm{E}}\rangle\,\theta$: a direct consequence of the inductive hypothesis.
– $\psi = [A]\theta$, $\psi = \langle\mathrm{A}\rangle\,\theta$, $\psi = [\overline{A}]\theta$ or $\psi = \langle\overline{\mathrm{A}}\rangle\,\theta$. Since $u \star \overline{\pi}^j$ and $u \star \overline{\rho}^{H(j)}$ start at the same state and lead to the same state (by Properties 2 and 3 of H), the result trivially follows. This concludes the proof of the claim.

We have proved that $\mathcal{K}, \overline{\pi} \models \varphi$, with $|\pi| < |\rho|$. Now, if $|\pi| \leq |W| \cdot (|\varphi| + 1)^2$, the thesis is proved, otherwise we can iterate the above contraction a finite number of times, until the bound is achieved. □

Now, by exploiting the polynomial-size model-track property stated by Theorem 2, it is easy to define a **PSPACE** model checking algorithm for $\mathsf{A}\overline{\mathsf{A}}\mathsf{E}\overline{\mathsf{E}}$ formulas, and a **co-NP** model checking algorithm for E formulas.

Algorithm 1 $\texttt{ModCheck}(\mathcal{K}, \psi)$

1: **for** all initial $\tilde{\rho} \in \mathrm{Trk}_{\mathcal{K}}$ s.t. $|\tilde{\rho}| \leq |W| \cdot (2|\psi|+3)^2$ **do**
2: **if** $\texttt{Check}(\mathcal{K}, \psi, \tilde{\rho}) = 0$ **then**
3: **return** 0: "$\mathcal{K}, \tilde{\rho} \not\models \psi$" ◁ *Counterexample found*
4: **return** 1: "$\mathcal{K} \models \psi$"

The main model checking procedure for $\mathsf{A}\overline{\mathsf{A}}\mathsf{E}\overline{\mathsf{E}}$ formulas is $\texttt{ModCheck}(\mathcal{K}, \psi)$ (Algorithm 1). All the initial tracks $\tilde{\rho}$, obtained by visiting the unravelling of $\mathcal{K}$ from w_0 up to depth $|W| \cdot (2|\psi| + 3)^2$, are checked w.r. to ψ by the function $\texttt{Check}(\mathcal{K}, \psi, \tilde{\rho})$ (Algorithm 2)—which decides whether $\mathcal{K}, \tilde{\rho} \models \psi$ by basically calling itself recursively on the sub-formulas of ψ and unravelling again $\mathcal{K}$—until either some initial track is found that does not model ψ or all of them model ψ (and thus $\mathcal{K} \models \psi$).

Notice that the for-loop at the first line considers all initial tracks of length at most $|W| \cdot (2|\psi| + 3)^2 \geq |W| \cdot (|NNF(\neg\psi)| + 1)^2$. The reason is that in the soundness/completeness proof of the algorithm, we need to consider the NNF of $\neg\psi$, and we exploit the polynomial bound of Theorem 2 applied to such a form.

The next theorem states soundness and completeness of the presented procedures (for the proof, see Appendix A and B).

Algorithm 2 $\texttt{Check}(\mathcal{K}, \psi, \tilde{\rho})$

1: **if** $\psi = p$, for $p \in \mathcal{AP}$ **then**
2: **if** $p \in \bigcap_{s \in \mathrm{states}(\tilde{\rho})} \mu(s)$ **then**
3: **return** 1 **else return** 0
4: **else if** $\psi = \varphi_1 \wedge \varphi_2$ **then**
5: **if** $\texttt{Check}(\mathcal{K}, \varphi_1, \tilde{\rho}) = 0$ **then**
6: **return** 0
7: **else**
8: **return** $\texttt{Check}(\mathcal{K}, \varphi_2, \tilde{\rho})$
9: **else if** $\psi = \langle\mathrm{A}\rangle\,\varphi$ **then**
10: **for** all $\rho \in \mathrm{Trk}_{\mathcal{K}}$ such that $\mathrm{fst}(\rho) = \mathrm{lst}(\tilde{\rho})$, and $|\rho| \leq |W| \cdot (2|\varphi| + 1)^2$ **do**
11: **if** $\texttt{Check}(\mathcal{K}, \varphi, \rho) = 1$ **then**
12: **return** 1
13: **return** 0
14: **else if** $\psi = \langle\mathrm{E}\rangle\,\varphi$ **then**
15: **for** each $\overline{\rho}$ suffix of $\tilde{\rho}$ **do**
16: **if** $\texttt{Check}(\mathcal{K}, \varphi, \overline{\rho}) = 1$ **then**
17: **return** 1
18: **return** 0
19: **else if** $\psi = \langle\overline{\mathrm{E}}\rangle\,\varphi$ **then**
20: **for** all $\rho \in \mathrm{Trk}_{\mathcal{K}}$ such that $\mathrm{lst}(\rho) = \mathrm{fst}(\tilde{\rho})$, and $2 \leq |\rho| \leq |W| \cdot (2|\varphi| + 1)^2$ **do**
21: **if** $\texttt{Check}(\mathcal{K}, \varphi, \rho \star \tilde{\rho}) = 1$ **then**
22: **return** 1
23: **return** 0
24: **else if** $\psi = \neg\varphi$ **then**
25: **return** $1 - \texttt{Check}(\mathcal{K}, \varphi, \tilde{\rho})$
26: ... ◁ $\psi = \langle\overline{\mathrm{A}}\rangle\,\varphi$ *is analogous to* $\psi = \langle\mathrm{A}\rangle\,\varphi$

Theorem 3. *Let ψ be an $\mathsf{A\overline{A}E\overline{E}}$ formula and $\mathcal{K}$ be a Kripke structure. Then, (i) $\texttt{ModCheck}(\mathcal{K}, \psi) = 1$ if and only if $\mathcal{K} \models \psi$; (ii) for any track $\tilde{\rho} \in \mathrm{Trk}_{\mathcal{K}}$, $\texttt{Check}(\mathcal{K}, \psi, \tilde{\rho}) = 1$ if and only if $\mathcal{K}, \tilde{\rho} \models \psi$.*

The given procedures require *polynomial working space*, since:

- `ModCheck` needs to store only a track no longer than $|W| \cdot (2|\psi| + 3)^2$ (obviously, many tracks are generated while visiting the unravelling of $\mathcal{K}$, but only one at a time needs to be stored);
- every recursive call to `Check` (possibly) needs space for a track no longer than $|W| \cdot (2|\varphi| + 1)^2$, where φ is a subformula of ψ such that $|\varphi| = |\psi| - 1$;
- at most 1 call to `ModCheck` and $|\psi|$ calls to `Check` can be jointly active.

Therefore, the maximum space needed by the given algorithms is $(|\psi| + 1) \cdot O(\log |W|) \cdot (|W| \cdot (2|\psi| + 3)^2)$ bits, where $O(\log |W|)$ bits are needed to represent a state of $\mathcal{K}$.

Corollary 1. *The model checking problem for $\mathsf{A\overline{A}E\overline{E}}$ formulas over finite Kripke structures is* **PSPACE***-complete.*

Proof. The **PSPACE**-hardness immediately follows from that of the fragment $\mathsf{\overline{E}}$, as proved in [22]. □

By means of simple modifications to the proposed procedures, it is possible to prove the following corollary. See Appendix C for a detailed explanation.

Corollary 2. *The model checking problem for E formulas over finite Kripke structures is* **co-NP***-complete.*

Proof (Sketch). It is easy to see that checking an E formula over a given track can be done in deterministic polynomial time in the size of the track and of the formula. Moreover, by Theorem 2, one can restrict to non-deterministically guessing a possible counterexample (i.e., an initial track *not* satisfying the input formula ψ) of length at most $|W| \cdot (|NNF(\neg\psi)| + 1)^2$. If a counterexample can be found, $\mathcal{K} \not\models \psi$. It follows that the model checking problem for E is in **co-NP**.

Finally, **co-NP**-hardness immediately follows from that of Prop [20]. □

5 Conclusions

In this paper, we have sharpened the border between good and bad fragments of Halpern and Shoham's modal logic of time intervals with respect to model checking. On the one hand, we have shown that the presence of both modality $\langle \mathrm{B} \rangle$ and modality $\langle \mathrm{E} \rangle$ suffices for a fragment to be **EXPSPACE**-hard. This lower bound immediately propagates to full HS. On the other hand, we have studied two well-behaved, **PSPACE**-complete fragments, $\mathsf{A\overline{A}E\overline{E}}$ and $\mathsf{A\overline{A}B\overline{B}}$, which are quite promising from the point of view of applications.

The fragment $\mathsf{A\overline{A}B\overline{B}E\overline{E}}$ (as well as the symmetric fragment $\mathsf{A\overline{A}E\overline{B}E\overline{E}}$), investigated in [21], still lies somehow across the border between good and bad fragments, as it is situated in between **EXPSPACE** and **PSPACE**. One possibility

for $\mathsf{A\overline{A}B\overline{BE}}$ is to be **PSPACE**-complete—which would mean that $\langle\overline{\mathrm{E}}\rangle$ does not add complexity to $\mathsf{A\overline{A}B\overline{B}}$, and analogously $\langle\mathrm{B}\rangle$ to $\mathsf{A\overline{ABE}}$. Another possibility is that the presence of both $\langle\mathrm{B}\rangle$ and $\langle\overline{\mathrm{E}}\rangle$ causes a significant blow-up in complexity. A larger complexity gap is the one for full HS: we have shown it to be **EXPSPACE**-hard, but the only known upper bound is non-elementary. In our future work, we will definitely come back to both $\mathsf{A\overline{A}B\overline{BE}}$ and full HS.

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A Proof of soundness/completeness of Algorithm 2

Lemma 1. *Let ψ be an $\mathsf{A\overline{A}E\overline{E}}$ formula, $\mathcal{K}$ be a Kripke structure, and $\tilde{\rho} \in \mathrm{Trk}_{\mathcal{K}}$. Then, $\texttt{Check}(\mathcal{K}, \psi, \tilde{\rho}) = 1$ if and only if $\mathcal{K}, \tilde{\rho} \models \psi$.*

Proof. The proof is by induction on the structure of ψ. (Base cases). The case in which $\psi = p$, for $p \in \mathcal{AP}$, follows from the definition. (Inductive cases). The cases in which $\psi = \neg\varphi$ and $\psi = \varphi_1 \wedge \varphi_2$ are also trivial and thus omitted. We focus on the remaining cases.

- $\psi = \langle \mathrm{A} \rangle \varphi$. If $\mathcal{K}, \tilde{\rho} \models \psi$, then there exists a track $\rho \in \mathrm{Trk}_{\mathcal{K}}$ such that $\mathrm{lst}(\tilde{\rho}) = \mathrm{fst}(\rho)$ and $\mathcal{K}, \rho \models \varphi$. By Theorem 2, there exists a track $\pi \in \mathrm{Trk}_{\mathcal{K}}$, with $|\pi| \leq |W| \cdot (|\varphi'| + 1)^2$ and $\mathrm{fst}(\pi) = \mathrm{fst}(\rho)(= \mathrm{lst}(\tilde{\rho}))$, such that $\mathcal{K}, \pi \models \varphi'$, where φ' is the NNF of φ. Thus, being $|\pi| \leq |W| \cdot (2|\varphi| + 1)^2$, such track π is considered in the for-loop at line 10. By the inductive hypothesis, $\texttt{Check}(\mathcal{K}, \varphi, \pi) = 1$ and thus $\texttt{Check}(\mathcal{K}, \psi, \tilde{\rho}) = 1$.
  Vice versa, if $\texttt{Check}(\mathcal{K}, \psi, \tilde{\rho}) = 1$, then there exists a track $\rho \in \mathrm{Trk}_{\mathcal{K}}$ such that $\mathrm{lst}(\tilde{\rho}) = \mathrm{fst}(\rho)$ for which $\texttt{Check}(\mathcal{K}, \varphi, \rho) = 1$. By the inductive hypothesis, $\mathcal{K}, \rho \models \varphi$, hence $\mathcal{K}, \tilde{\rho} \models \psi$.

– $\psi = \langle\overline{\mathrm{A}}\rangle\,\varphi$ is analogous to the previous case.
– $\psi = \langle\mathrm{E}\rangle\,\varphi$. If $\mathcal{K}, \tilde{\rho} \models \psi$, there exists a track $\rho \in \mathrm{Suff}(\tilde{\rho})$ such that $\mathcal{K}, \rho \models \varphi$. By the inductive hypothesis, $\texttt{Check}(\mathcal{K}, \varphi, \rho) = 1$. Since all the suffixes of $\tilde{\rho}$ are checked, $\texttt{Check}(\mathcal{K}, \psi, \tilde{\rho}) = 1$.
Vice versa, if $\texttt{Check}(\mathcal{K}, \psi, \tilde{\rho}) = 1$, then for some $\rho \in \mathrm{Suff}(\tilde{\rho})$, it holds that $\texttt{Check}(\mathcal{K}, \varphi, \rho) = 1$. By the inductive hypothesis $\mathcal{K}, \rho \models \varphi$, hence $\mathcal{K}, \tilde{\rho} \models \psi$.
– $\psi = \langle\overline{\mathrm{E}}\rangle\,\varphi$. If $\mathcal{K}, \tilde{\rho} \models \psi$, then there exists a track $\rho \in \mathrm{Trk}_{\mathcal{K}}$, with $|\rho| \geq 2$, such that $\mathcal{K}, \rho \star \tilde{\rho} \models \varphi$. By Theorem 2, there exists a track $\pi \in \mathrm{Trk}_{\mathcal{K}}$ induced by ρ, with $|\pi| \leq |W| \cdot (|\varphi'| + 1)^2$, such that $\mathcal{K}, \pi \star \tilde{\rho} \models \varphi'$, where φ' is the NNF of φ. Such track π is considered in the for-loop at line 20, since $|\pi| \leq |W| \cdot (2|\varphi| + 1)^2$ and $|\pi| \geq 2$ as it is induced by ρ. By the inductive hypothesis, $\texttt{Check}(\mathcal{K}, \varphi, \pi \star \tilde{\rho}) = 1$, hence $\texttt{Check}(\mathcal{K}, \psi, \tilde{\rho}) = 1$.
Vice versa, if $\texttt{Check}(\mathcal{K}, \psi, \tilde{\rho}) = 1$, there exists a track $\rho \in \mathrm{Trk}_{\mathcal{K}}$, with $|\rho| \geq 2$, such that $\texttt{Check}(\mathcal{K}, \varphi, \rho \star \tilde{\rho}) = 1$. By the inductive hypothesis, $\mathcal{K}, \rho \star \tilde{\rho} \models \varphi$, hence $\mathcal{K}, \tilde{\rho} \models \psi$. □

B Proof of soundness/completeness of Algorithm 1

Theorem 4. *Let ψ be an $\mathsf{A\overline{A}E\overline{E}}$ formula and $\mathcal{K}$ be a Kripke structure. Then, $\texttt{ModCheck}(\mathcal{K}, \psi) = 1$ if and only if $\mathcal{K} \models \psi$.*

Proof. ($\Leftarrow$) If $\mathcal{K} \models \psi$, then, for all initial tracks $\rho \in \mathrm{Trk}_{\mathcal{K}}$, we have that $\mathcal{K}, \rho \models \psi$. By Lemma 1, it follows that $\texttt{Check}(\mathcal{K}, \psi, \rho) = 1$. Now, the for-loop at line 1 considers a subset of the initial tracks. This implies that $\texttt{ModCheck}(\mathcal{K}, \psi) = 1$.

($\Rightarrow$) If $\texttt{ModCheck}(\mathcal{K}, \psi) = 1$, then, for any initial track ρ considered by the for-loop at line 1, that is, with $|\rho| \leq |W| \cdot (2|\psi|+3)^2$, it holds that $\texttt{Check}(\mathcal{K}, \psi, \rho) = 1$. Let us assume by contradiction that $\mathcal{K} \not\models \psi$, that is, there exists an initial track $\overline{\rho} \in \mathrm{Trk}_{\mathcal{K}}$ such that $\mathcal{K}, \overline{\rho} \models \neg\psi$, or, equivalently, $\mathcal{K}, \overline{\rho} \models \overline{\psi}$, where $\overline{\psi}$ is the NNF of $\neg\psi$. Thus, by Theorem 2, there exists an initial track $\tilde{\rho}$ with $|\tilde{\rho}| \leq |W| \cdot (|\overline{\psi}| + 1)^2 \leq |W| \cdot (2|\psi| + 3)^2$, such that $\mathcal{K}, \tilde{\rho} \models \overline{\psi}$, namely, $\mathcal{K}, \tilde{\rho} \not\models \psi$. By Lemma 1, it holds that $\texttt{Check}(\mathcal{K}, \psi, \tilde{\rho}) = 0$. This leads to a contradiction. Therefore $\mathcal{K} \models \psi$. □

C Model checking algorithm for E formulas

Here we describe a model checking algorithm for the fragment E, which, in its turn, heavily rests on the polynomial-size model-track property.
$\texttt{CounterExE}(\mathcal{K}, \psi)$ is a non-deterministic procedure (Algorithm 3) which searches for counterexamples to the input E formula ψ, that is, initial tracks satisfying $\neg\psi$. If such a counterexample is found, clearly $\mathcal{K} \not\models \psi$. First, the procedure generates in a *non-deterministic way* an initial track $\tilde{\rho}$, of length at most $|W| \cdot (2|\psi| + 3)^2$, by means of $\texttt{A\_track}(\mathcal{K}, w_0, |\psi|)$. Then, $\texttt{CheckE}(\mathcal{K}, \psi, \tilde{\rho})$ (Algorithm 4) evaluates ψ over the track $\tilde{\rho}$ in a *deterministic way*. If CheckE returns $\bot$, a counterexample has been found and CounterExE returns **Yes** (thus the non-deterministic computation of the algorithm is successful). Otherwise, it returns **No** (the computation fails).

Algorithm 3 `CounterExE`$(\mathcal{K}, \psi)$

1: $\tilde{\rho} \leftarrow$ `A_track`$(\mathcal{K}, w_0, |\psi|)$ ◁ *a track of* $\mathcal{K}$ *from* w_0 *of length* $\leq |W| \cdot (2|\psi| + 3)^2$
2: **if** `CheckE`$(\mathcal{K}, \psi, \tilde{\rho}) = \bot$ **then**
3: **return Yes**: "$\mathcal{K}, \tilde{\rho} \not\models \psi$" ◁ *Counterexample found*
4: **else**
5: **return No**: "$\mathcal{K}, \tilde{\rho} \models \psi$" ◁ *Counterexample* not *found*

Algorithm 4 `CheckE`$(\mathcal{K}, \chi, \tilde{\rho})$

1: $T \leftarrow$ `New_Table`$(|\chi|, |\tilde{\rho}|)$
2: **for** all subformulas φ of χ by increasing length **do**
3: **if** $\varphi = p$, for $p \in \mathcal{AP}$ **then**
4: $T[p, |\tilde{\rho}|] \leftarrow p \in \mu(\mathrm{lst}(\tilde{\rho}))$
5: **for** $i = |\tilde{\rho}| - 1, \cdots, 1$ **do**
6: $T[p, i] \leftarrow T[p, i + 1]$ and $p \in \mu(\tilde{\rho}(i))$
7: **else if** $\varphi = \neg\varphi_1$ **then**
8: **for** $i = |\tilde{\rho}|, \cdots, 1$ **do**
9: $T[\varphi, i] \leftarrow$ not $T[\varphi_1, i]$
10: **else if** $\varphi = \varphi_1 \wedge \varphi_2$ **then**
11: **for** $i = |\tilde{\rho}|, \cdots, 1$ **do**
12: $T[\varphi, i] \leftarrow T[\varphi_1, i]$ and $T[\varphi_2, i]$
13: **else if** $\varphi = \langle \mathrm{E} \rangle \varphi_1$ **then**
14: $T[\varphi, |\tilde{\rho}|] \leftarrow \bot$
15: **for** $i = |\tilde{\rho}| - 1, \cdots, 1$ **do**
16: $T[\varphi, i] \leftarrow T[\varphi, i + 1]$ or $T[\varphi_1, i + 1]$
17: **return** $T[\chi, 1]$

As for `CheckE`, it clearly holds that `CheckE`$(\mathcal{K}, \chi, \tilde{\rho}) = \top \iff \mathcal{K}, \tilde{\rho} \models \chi$, for any E formula χ. This procedure scans all the sub-formulas φ of the input χ by increasing length, and annotates in the Boolean entry $T[\varphi, i]$ (for $1 \leq i \leq |\tilde{\rho}|$) of the table T whether $\mathcal{K}, \tilde{\rho}^i \models \varphi$ or not. At line 2, when the sub-formula φ of χ is being considered, it holds that for all other sub-formulas ξ processed in some previous iteration, $T[\xi, i] = \top \iff \mathcal{K}, \tilde{\rho}^i \models \xi$.

We now briefly prove soundness and completeness of `CounterExE`. On the one hand, if `CounterExE`$(\mathcal{K}, \psi)$ has a successful computation, then there exists an initial track $\tilde{\rho}$, such that `CheckE`$(\mathcal{K}, \psi, \tilde{\rho}) = \bot$. This means that $\mathcal{K}, \tilde{\rho} \not\models \psi$, and thus $\mathcal{K} \not\models \psi$. On the other hand, if $\mathcal{K} \not\models \psi$, then there exists an initial track ρ such that $\mathcal{K}, \rho \not\models \psi$. By Theorem 2, there exists an initial track $\tilde{\rho}$, of length at most $|W| \cdot (|\psi'| + 1)^2 \leq |W| \cdot (2|\psi| + 3)^2$, such that $\mathcal{K}, \tilde{\rho} \models \psi'$, where ψ' is the NNF of $\neg\psi$. Now, some non-deterministic instance of `A_track`$(\mathcal{K}, w_0, |\psi|)$ generates exactly such $\tilde{\rho}$, being $|\tilde{\rho}| \leq |W| \cdot (2|\psi| + 3)^2$. Moreover, `CheckE`$(\mathcal{K}, \psi, \tilde{\rho}) = \bot$, and thus `CounterExE`$(\mathcal{K}, \psi)$ has a successful computation.

`CounterExE`$(\mathcal{K}, \psi)$ is in **NP**, as the generated track(s) $\tilde{\rho}$ has (have) a length polynomial in $|W|$ and $|\psi|$, and can thus be calculated in polynomial time. Subsequently, `CheckE` performs a polynomial number of steps, since all it has to do is filling in the table T, which features $|\psi| \cdot |\tilde{\rho}|$ entries.

Corollary 2. *The model checking problem for* E *formulas over finite Kripke structures is* **co-NP***-complete.*

Proof. Membership of the problem to **co-NP** follows as `CounterExE`$(\mathcal{K}, \psi)$ has a successful computation if and only if $\mathcal{K} \not\models \psi$, and such a procedure runs in (non-deterministic) polynomial time. The **co-NP**-hardness derives immediately from that of the purely propositional fragment of HS, Prop, as proved in [20]. □