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# Decidability of the interval temporal logic $\mathsf{A\bar{A}B\bar{B}}$ over the rationals

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**Abstract.** The classification of the fragments of Halpern and Shoham's logic with respect to decidability/undecidability of the satisfiability problem is now very close to the end. We settle one of the few remaining questions concerning the fragment $\mathsf{A\bar{A}B\bar{B}}$, which comprises Allen's interval relations "meets" and "begins" and their symmetric versions. We already proved that $\mathsf{A\bar{A}B\bar{B}}$ is decidable over the class of all finite linear orders and undecidable over ordered domains isomorphic to $\mathbb{N}$. In this paper, we first show that $\mathsf{A\bar{A}B\bar{B}}$ is undecidable over $\mathbb{R}$ and over the class of all Dedekind-complete linear orders. We then prove that the logic is decidable over $\mathbb{Q}$ and over the class of all linear orders.

## 1 Introduction

Even though it has been authoritatively and repeatedly claimed that interval-based formalisms are the most appropriate ones for a variety of application domains, e.g., [6], until very recently interval temporal logics were a largely unexplored land. There are at least two explanations for such a situation: computational complexity and technical difficulty. On the one hand, the seminal work by Halpern and Shoham on the interval logic of Allen's interval relations (HS for short) showed that such a logic is highly undecidable over all meaningful classes of linear orders [5], and ten years later Lodaya proved that a restricted fragment of it, denoted $\mathsf{BE}$, featuring only two modalities (those for Allen's relations *begins* and *ends*), suffices for undecidability [7]. On the other hand, formulas of interval temporal logics express properties of pairs of time points rather than of single time points, and are evaluated as sets of such pairs, that is, binary relations. As a consequence, there is no reduction of the satisfiability/validity in interval logics to monadic second-order logic, and thus Rabin's theorem (the standard proof machinery) is not applicable here.

In the last decade, a systematic investigation of HS fragments has been carried out. Their classification with respect to the decidability/undecidability of their satisfiability problem is now very close to the end. The outcome of the analysis is that undecidability rules over HS fragments [1, 8], but some meaningful exceptions exist [2, 3, 4, 10, 11]. While setting the status of most and least expressive interval logics is relatively straightforward, e.g., undecidability of full HS can be shown by a reduction from the non-halting problem for Turing machines, decidability of the logic of Allen's relations *begins* and *begun by* $\mathsf{B\bar{B}}$ can be proved by a reduction to the (point-based) linear temporal logic of future and past, dealing with those fragments that lie on the marginal land between decidability and undecidability is much more difficult. (Un)decidability of HS fragments depends on two factors: their set of interval modalities and the class of linear orders over which they are interpreted. While the first one is fairly obvious, the second one is definitively less immediate. Some HS fragments behave the same over all classes of linear orders. This is the case with the logic of temporal neighbourhood $\mathsf{A\bar{A}}$, which is NEXPTIME-complete over all relevant classes of linear orders [3]. A real character is, on the contrary, the temporal logic of sub-intervals $\mathsf{D}$: its satisfiability problem is PSPACE-complete over the class of dense linear orders [2] and undecidable over the classes of finite and discrete linear orders [8] (it is still unknown over the class of all linear orders).

In this paper, we focus our attention on the satisfiability problem for the logic $\mathsf{A\bar{A}B\bar{B}}$, which pairs the decidable fragments $\mathsf{A\bar{A}}$ and $\mathsf{B\bar{B}}$. In [11], we proved that the problem is decidable, but not primitive recursive, over finite linear orders, and undecidable over the natural numbers. Here, we first show that undecidability can be lifted to the temporal domain $\mathbb{R}$, as well as to the class of all Dedekind-complete linear orders. Then, we consider the order $\mathbb{Q}$. We devise two semi-decision procedures: the first one terminates if and only if the input formula is unsatisfiable over $\mathbb{Q}$, while the second one terminates if and only if the input formula is satisfiable over $\mathbb{Q}$. Running the two procedures in parallel gives a decision algorithm for $\mathsf{A\bar{A}B\bar{B}}$ over $\mathbb{Q}$. We conclude the paper by showing that decidability over the class of all linear orders follows from that over $\mathbb{Q}$. All proofs are given in the Appendix.

## 2 The logic

We begin by introducing the logic $\mathsf{A\bar{A}B\bar{B}}$. Let $\Sigma$ be a set of proposition letters. The logic $\mathsf{A\bar{A}B\bar{B}}$ consists of formulas built up from letters in $\Sigma$ using the Boolean connectives $\neg$ and $\vee$ and the unary modalities $\langle \mathsf{A} \rangle$, $\langle \bar{\mathsf{A}} \rangle$, $\langle \mathsf{B} \rangle$, and $\langle \bar{\mathsf{B}} \rangle$. We will often make use of shorthands like $\varphi_1 \wedge \varphi_2 = \neg(\neg\varphi_1 \vee \neg\varphi_2)$, $[\mathsf{A}]\varphi = \neg\langle \mathsf{A} \rangle\neg\varphi$, $[\mathsf{B}]\varphi = \neg\langle \mathsf{B} \rangle\neg\varphi$, $\mathsf{true} = a \vee \neg a$, and $\mathsf{false} = a \wedge \neg a$, for $a \in \Sigma$.

To define the semantics of $\mathsf{A\bar{A}B\bar{B}}$ formulas, we consider a linear order $\mathbb{D} = (D, <)$, called *temporal domain*, and we denote by $\mathbb{I}_{\mathbb{D}}$ the set of all closed intervals $[x, y]$ over $\mathbb{D}$, with $x \le y$. We call *interval structure* any Kripke structure of the form $\mathcal{I} = (\mathbb{I}_{\mathbb{D}}, \sigma, A, \bar{A}, B, \bar{B})$, where $\sigma : \mathbb{I}_{\mathbb{D}} \to \mathscr{P}(\Sigma)$ is a function mapping intervals to sets of proposition letters and $A$, $\bar{A}$, $B$, and $\bar{B}$ are the Allen's relations "meet", "met by", "begun by", and "begins", which are defined as follows: $[x, y]\ A\ [x', y']$ iff $y = x'$, $[x, y]\ \bar{A}\ [x', y']$ iff $x = y'$, $[x, y]\ B\ [x', y']$ iff $x = x' \wedge y' < y$, and $[x, y]\ \bar{B}\ [x', y']$ iff $x = x' \wedge y < y'$. Formulas are interpreted over a given interval structure $\mathcal{I} = (\mathbb{I}_{\mathbb{D}}, \sigma, A, \bar{A}, B, \bar{B})$ and a given initial interval $I \in \mathbb{I}_{\mathbb{D}}$ in the natural way, as follows: $\mathcal{I}, I \vDash a$ iff $a \in \sigma(I)$, $\mathcal{I}, I \vDash \neg\varphi$ iff $\mathcal{I}, I \nvDash \varphi$, $\mathcal{I}, I \vDash \varphi_1 \vee \varphi_2$ iff $\mathcal{I}, I \vDash \varphi_1$ or $\mathcal{I}, I \vDash \varphi_2$, and, most importantly, for all relations $R \in \{A, \bar{A}, B, \bar{B}\}$,

$$\mathcal{I}, I \vDash \langle \mathsf{R} \rangle\varphi \qquad \text{iff} \qquad \text{there is } J \in \mathbb{I}_{\mathbb{D}} \text{ such that } I\ R\ J \text{ and } \mathcal{I}, J \vDash \varphi.$$

We say that a formula $\varphi$ is *satisfiable* over a class $\mathscr{C}$ of interval structures if $\mathcal{I}, I \vDash \varphi$ for some $\mathcal{I} = (\mathbb{I}_{\mathbb{D}}, \sigma, A, \bar{A}, B, \bar{B})$ in $\mathscr{C}$ and some interval $I \in \mathbb{I}_{\mathbb{D}}$.

For example, the formula $[\mathsf{B}]\mathsf{false}$ (hereafter abbreviated $\pi$) hold over all and only the *singleton* intervals $[x, x]$. Similarly, the formula $[\mathsf{B}][\mathsf{B}]\mathsf{false}$ (abbreviated $\mathsf{unit}$) holds over the *unit-length* intervals of a discrete order, e.g. over the intervals of $\mathbb{Z}$ of the form $[x, x+1]$. The formula $[\bar{\mathsf{A}}][\bar{\mathsf{A}}][\mathsf{A}][\mathsf{A}]\varphi$ ($[\mathsf{G}]\varphi$ for short) forces $\varphi$ to hold universally, that is, over all intervals. The formula $[\mathsf{G}]\,(\,\neg\pi \rightarrow \langle\mathsf{B}\rangle\neg\pi\,)$ ($\varphi_{\mathsf{dense}}$ for short) holds over all and only the interval structures with a dense domain, e.g., the order $\mathbb{Q}$ of the rationals.

**Logical types.** We now introduce basic terminology and notation that are common in the temporal logic setting. The *closure* of a formula $\varphi$ is defined as the set $\mathsf{closure}(\varphi)$ of all sub-formulas of $\varphi$ and all their negations (we identify $\neg\neg\psi$ with $\psi$, $\neg\langle\mathsf{A}\rangle\psi$ with $[\mathsf{A}]\neg\psi$, etc.). For a technical reason that will be clear soon, we also introduce the *extended closure* of $\varphi$, denoted $\mathsf{closure}^{+}(\varphi)$, that extends $\mathsf{closure}(\varphi)$ by adding all formulas of the form $\langle\mathsf{R}\rangle\psi$ and $[R]\psi$, with $R \in \{A, \bar{A}, B, \bar{B}\}$ and $\psi \in \mathsf{closure}(\varphi)$.

Let $\mathcal{I} = (\mathbb{I}_{\mathbb{D}}, \sigma, A, \bar{A}, B, \bar{B})$ be an interval structure. We associate with each interval $I \in \mathbb{I}_{\mathbb{D}}$ its *$\varphi$-type* $\mathsf{type}^{\varphi}_{\mathcal{I}}(I)$, defined as the set of all formulas $\psi \in \mathsf{closure}^{+}(\varphi)$ such that $\mathcal{I}, I \vDash \psi$ (when no confusion arises, we omit the parameters $\mathcal{I}$ and $\varphi$). A particular role will be played by those types $F$ that contain the subformula $[\mathsf{B}]\mathsf{false}$, which are necessarily associated with singleton intervals. When no interval structure is given, we can still try to capture the concept of type by means of a maximal "locally consistent" subset of $\mathsf{closure}^{+}(\varphi)$. Formally, we call *$\varphi$-atom* any set $F \subseteq \mathsf{closure}^{+}(\varphi)$ such that (i) $\psi \in F$ iff $\neg\psi \notin F$, for all $\psi \in \mathsf{closure}^{+}(\varphi)$, (ii) $\psi \in F$ iff $\psi_1 \in F$ or $\psi_2 \in F$, for all $\psi = \psi_1 \vee \psi_2 \in \mathsf{closure}^{+}(\varphi)$, (iii) if $[\mathsf{B}]\mathsf{false} \in F$ and $\psi \in F$, then $\langle\mathsf{A}\rangle\psi \in F$ and $\langle\bar{\mathsf{A}}\rangle\psi \in F$, for all $\psi \in \mathsf{closure}(\varphi)$, (iv) if $[\mathsf{B}]\mathsf{false} \in F$ and $\langle\mathsf{A}\rangle\psi \in F$, then $\psi \in F$ or $\langle\bar{\mathsf{B}}\rangle\psi \in F$, for all $\psi \in \mathsf{closure}(\varphi)$. We call *$\pi$-atoms* those atoms that contain the formula $[\mathsf{B}]\mathsf{false}$, which are thus candidate types of singleton intervals. We denote by $\mathsf{atoms}(\varphi)$ the set of all $\varphi$-atoms.

Given an atom $F$ and a relation $R \in \{A, \bar{A}, B, \bar{B}\}$, we let $\mathsf{req}_R(F)$ be the set of *requests of $F$ along direction $R$*, namely, the formulas $\psi \in \mathsf{closure}(\varphi)$ such that $\langle\mathsf{R}\rangle\psi \in F$. Similarly, we let $\mathsf{obs}(F)$ be the set of *observables of $F$*, namely, the formulas $\psi \in F \cap \mathsf{closure}(\varphi)$ – intuitively, the observables of $F$ are those formulas $\psi \in F$ that fulfil requests of the form $\langle\mathsf{R}\rangle\psi$ from other atoms. Note that, for all $\pi$-atoms $F$, we have $\mathsf{req}_A(F) = \mathsf{obs}(F) \cup \mathsf{req}_{\bar{B}}(F)$ and $\mathsf{req}_{\bar{A}}(F) \supseteq \mathsf{obs}(F)$.

**Compass structures.** Formulas of interval temporal logics can be equivalently interpreted over the so-called compass structures [14]. These structures can be seen as two-dimensional spaces in which points are labelled with complete logical types (atoms). Such an alternative interpretation exploits the existence of a natural bijection between the intervals $I = [x, y]$ over a temporal domain $\mathbb{D}$ and the points $p = (x, y)$ in the $\mathbb{D}\times\mathbb{D}$ grid such that $x \le y$. It is convenient to introduce a *dummy atom* $\varnothing$, distinct from all other atoms, and assume that it labels all and only the points $(x, y)$ such that $x > y$, which do not correspond to intervals. We fix the convention that $\mathsf{obs}(\varnothing) = \varnothing$ and $\mathsf{req}_R(\varnothing) = \varnothing$ for all $R \in \{A, \bar{A}, B, \bar{B}\}$.

Formally, a *compass $\varphi$-structure over a linear order $\mathbb{D}$* is a labelled grid $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$, where the function $\tau : \mathbb{D} \times \mathbb{D} \rightarrow \mathsf{atoms}(\varphi) \uplus \{\varnothing\}$ maps any point $(x, y)$ to either a $\varphi$-atom (if $x \le y$) or the dummy atom $\varnothing$ (if $x > y$).

We observe that Allen's relations over intervals have analogue relations over points. Figure 1 gives a geometric interpretation of relations $A, \bar{A}, B, \bar{B}$ (by a slight abuse of notation, we use the same letters for the corresponding relations over the points of a compass structure). Thanks to such an interpretation, any interval structure $\mathcal{I}$ can be converted to a compass one $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$ by simply letting $\tau(x,y) = \mathsf{type}([x,y])$ for all $x \leq y \in \mathbb{D}$. The converse, however, is not true in general, as the atoms associated with points in a compass structure may be inconsistent with respect to the underlying geometrical interpretation of Allen's relations. To ease a correspondence between interval and compass structures, we enforce suitable *consistency conditions* on compass structures. For this, we introduce two relations over atoms $F, G$:

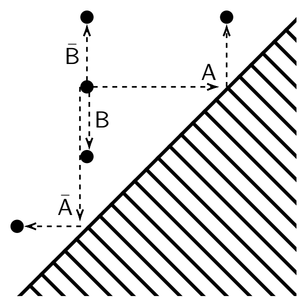


**Fig. 1.** Geometric interpretation of Allen's relations.

$F \uparrow G$ iff

$$\begin{cases} \mathsf{req}_{\bar{B}}(F) \supseteq \mathsf{obs}(G) \cup \mathsf{req}_{\bar{B}}(G) \\ \mathsf{req}_{B}(G) \supseteq \mathsf{obs}(F) \cup \mathsf{req}_{B}(F) \\ \mathsf{req}_{\bar{A}}(F) = \mathsf{req}_{\bar{A}}(G) \end{cases}$$

$F \nearrow G$ iff

$$\begin{cases} \mathsf{req}_{A}(F) = \mathsf{obs}(G) \cup \mathsf{req}_{B}(G) \cup \mathsf{req}_{\bar{B}}(G) \\ \mathsf{req}_{\bar{A}}(G) \supseteq \mathsf{obs}(F). \end{cases}$$

Note that the relation $\uparrow$ is transitive, while $\nearrow$ only satisfies $\nearrow \circ \uparrow \subseteq \nearrow$. Observe also that, for all interval structures $\mathcal{I}$ and all intervals $I, J$ in it, if $I \; \bar{B} \; J$ (resp., $I \; A \; J$), then $\mathsf{type}_{\mathcal{I}}(I) \uparrow \mathsf{type}_{\mathcal{I}}(J)$ (resp., $\mathsf{type}_{\mathcal{I}}(I) \nearrow \mathsf{type}_{\mathcal{I}}(J)$). Hereafter, we tacitly assume that every compass structure $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$ satisfies analogous consistency properties with respect to its atoms, namely, for all points $p = (x,y)$ and $q = (x', y')$ in $\mathbb{D} \times \mathbb{D}$, with $x \leq y$ and $x' \leq y'$, if $p \; \bar{B} \; q$ (resp., $p \; A \; q$), then $\tau(p) \uparrow \tau(q)$ (resp., $\tau(p) \nearrow \tau(q)$). In addition, we say that a request $\psi \in \mathsf{req}_R(\tau(p))$ of a point $p$ in a compass structure $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$ is *fulfilled* if there is another point $q$ such that $p \; R \; q$ and $\psi \in \mathsf{obs}(\tau(q))$ – in this case, we say that $q$ is a *witness of fulfilment of* $\psi$ *from* $p$. The compass structure $\mathcal{G}$ is said to be *globally fulfilling* if all requests of all its points are fulfilled.

We can now recall the standard correspondence between interval and compass structures (the proof is based on a simple induction on sub-formulas):

**Proposition 1** ([11])**.** Let $\varphi$ be an $\mathsf{A\bar{A}B\bar{B}}$ formula. For every globally fulfilling compass structure $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$, there is an interval structure $\mathcal{I} = (\mathbb{I}_{\mathbb{D}}, \sigma, A, \bar{A}, B, \bar{B})$ such that, for all $x \leq y \in \mathbb{D}$ and all $\psi \in \mathsf{closure}^{+}(\varphi)$, $\mathcal{I}, [x,y] \models \psi$ iff $\psi \in \tau(x,y)$.

In view of Proposition 1, the satisfiability problem for a given $\mathsf{A\bar{A}B\bar{B}}$ formula $\varphi$ reduces to the problem of deciding the existence of a globally fulfilling compass $\tilde{\varphi}$-structure $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$, with $\tilde{\varphi} = \langle \mathsf{G} \rangle \varphi$ ($\langle \mathsf{G} \rangle \varphi$ is a shorthand for $\neg [\mathsf{G}] \neg \varphi$), that features the observable $\tilde{\varphi}$ in every point, that is, $\tilde{\varphi} \in \mathsf{obs}(\tau(x,y))$ for all $x \leq y \in \mathbb{D}$.

## 3 Satisfiability over finite and Dedekind-complete orders

The satisfiability problem for $\mathsf{A\bar{A}B\bar{B}}$ was originally addressed in [11]. We first proved that $\mathsf{A\bar{A}B\bar{B}}$ is decidable if interpreted over finite linear orders, but not primitive recursive. The decidability result rests on a contraction method that, given a formula $\varphi$ and a finite compass structure satisfying $\varphi$, shows that, under suitable conditions, the compass structure can be reduced in size while preserving consistency and fulfilment properties. This leads to a non-deterministic procedure that decides whether $\varphi$ is satisfiable by exhaustively searching all contraction-free compass structures. The proof of termination relies on Dickson's lemma, while non-primitive recursiveness is proved via a reduction from the reachability problem for lossy counter machines [13]. Then, we showed that the problem becomes undecidable if we interpret $\mathsf{A\bar{A}B\bar{B}}$ over a temporal domain isomorphic to $\mathbb{N}$ (in fact, this is already the case with the proper fragment $\mathsf{A\bar{A}B}$). The proof is based on a reduction from an undecidable variant of the reachability problem for lossy counter machines, called structural termination [9], which consists of deciding whether a given lossy counter machine admits a halting computation starting from a given location and some arbitrary initial assignment for the counters. Due to an oversight, in [11] we claimed that such an undecidability result can be transferred to any class of linear orders in which $\mathbb{N}$ can be embedded. As a matter of fact, Dedekind completeness is a necessary condition. The following theorem properly states undecidability results for $\mathsf{A\bar{A}B}$.

**Theorem 1.** The satisfiability problem for $\mathsf{A\bar{A}B}$ interpreted over $\mathbb{N}$, $\mathbb{R}$, and the class of all Dedekind-complete linear orders is undecidable.

In view of the above theorem and the decidability results in [11], the satisfiability problem for $\mathsf{A\bar{A}B\bar{B}}$ over $\mathbb{Q}$, as well as over the class of all interval structures, remains open. In the next section, we will show that, quite surprisingly, both problems are decidable with non-primitive recursive complexity.

## 4 Satisfiability over the rationals and all linear orders

We begin by describing a fairly simple semi-decision procedure for the *unsatisfiability* of $\mathsf{A\bar{A}B\bar{B}}$ formulas over interval structures with a dense temporal domain. The crucial observation is that, whenever a formula $\varphi$ is unsatisfiable over $\mathbb{Q}$, this can be witnessed by a *finite* set of intervals with inconsistent requests. Based on this observation, one can enumerate all finite compass structures that witness $\varphi$ and are distinct up to isomorphism, following the partial order induced by the embedding relation (this relation is defined as an isomorphism between the smaller structure and the restriction of the larger structure to a suitable subset of its temporal domain). The only way the enumeration procedure can terminate is when no refinement is applicable: in this case, one proves that the input formula $\varphi$ is not satisfiable. Conversely, if the enumeration procedure does not terminate, then the formula $\varphi$ is satisfied by some compass structure that is obtained from the limit of an infinite series of refinements (suitable fairness conditions for the generated refinements guarantee that the temporal domain of the limit compass structure is isomorphic to $\mathbb{Q}$).

The rest of the section is devoted to finding a semi-decision procedure that receives an input formula $\varphi$ and terminates (successfully) iff $\varphi$ is satisfiable over an interval structure with a dense temporal domain. Differently from the previous procedure, this one is based on enumerating suitable *finite abstractions* of compass structures, which is far from being an easy task.

A first step consists of simplifying the consistency and fulfilment conditions. More precisely, we show how to turn them into more "local" constraints, so as to ease, later, the abstraction task. To this end, recall that the rational line is isomorphic to any countable dense ordering with neither a minimal element nor a maximal one. This means that, for the purpose of studying satisfiability over $\mathbb{Q}$, it does not matter if we consider interval structures over $\mathbb{Q}$ or over any subset of it that is dense and contains no extremal elements. Similarly, the complexity of the satisfiability problem does not change if we add minimal and maximal elements to the underlying temporal domain – for the sake of brevity, we call the resulting order a *dense order with endpoints*. Now, to turn the consistency and fulfilment conditions into local constraints, we decompose any dense order with endpoints $\mathbb{D}$ into some infinite, finitely-branching tree $T$ whose nodes represent pairs of elements of $\mathbb{D}$ of the form $s = (y_1, y_2)$, with $y_1 < y_2$, and whose edges connect nodes $(y_1, y_2) \in T$ to tuples of nodes $s_1 = (z_1, z_2)$, ..., $s_n = (z_n, z_{n+1})$, with $n \geq 2$ and $y_1 = z_1 < z_2 < \ldots < z_n < z_{n+1} = y_2$ (see Figure 2). Note that the domain $\mathbb{D}$ is not necessarily entirely covered by the time points that appear in the nodes of a decomposition $T$. Moreover, since all dense orders with endpoints are isomorphic, we will not be concerned with the coordinates of the nodes of $T$ and we will often overlook them in the constructions that follow.

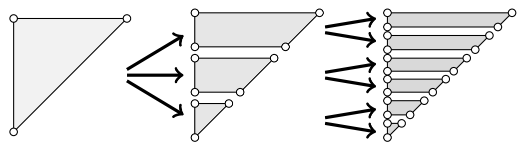

**Fig. 2.** Decomposition of a compass structure.

Using decompositions of temporal domains we can extract "horizontal slices" of a compass structure. More precisely, given a compass structure $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$ and a node $s = (y_1, y_2)$ of a decomposition $T$ of $\mathbb{D}$, we define the *slice of $\mathcal{G}$ in $s$* as the induced sub-structure $\mathcal{G}_s = (\mathbb{D} \times \{y_1, y_2\}, \tau)$. Intuitively, the slice $\mathcal{G}_s$ is obtained from $\mathcal{G}$ by selecting the rows with coordinates $y_1$ and $y_2$ and by restricting the labelling function $\tau$ to them (to reduce the notational overload, we denote such a restriction of the labelling function by $\tau$).

Below, we introduce suitable abstractions, called profiles, for the labels that can appear in a slice of a compass structure. Intuitively, for each slice $\mathcal{G}_s = (\mathbb{D} \times \{y_1, y_2\}, \tau)$ and each pair of atoms $(F, G)$, where possibly $F = \varnothing$ or both $F = \varnothing$ and $G = \varnothing$ (dummy atoms), we keep track of the *number* of coordinates $x \in \mathbb{D}$ such that $\tau(x, y_1) = F$ and $\tau(x, y_2) = G$. In particular, in these abstractions, we forget the occurrence order of the pairs of atoms along the $x$-axis. To this end, we make extensive use of *multisets*. Given a multiset $M$ and an element $e$ in $M$, we denote by $M(e)$ the *multiplicity*, that is, the number of occurrences, of $e$ in $M$, and we write $M(e) = \infty$ when $M$ contains infinitely many occurrences of $e$. We freely use set-theoretic notations with multisets. For example, we denote membership by $e \in M$, containment by $M \subseteq N$, etc. Moreover, given a multiset $M$

of $n$-tuples and a set $I \subseteq \{1, \dots, n\}$ of component indices, we denote by $M|_I$ the *projection* of $M$ onto $I$, that is, the multiset that contains exactly $\sum_{e|_I = f} M(e)$ occurrences of each $I$-tuple $f$ (note that the sum ranges over all $n$-tuples $e$ that coincide with $f$ on the components indexed in $I$). Note that, differently from set projections, projections of multisets are injective, as they send distinct occurrences of tuples to distinct occurrences of tuples. In particular, $|_I$ defines a *bijection* from multiset $M$ to multiset $M|_I$. Finally, we denote by $\mathsf{set}(M)$ the *support* of a multiset $M$, that is, the set of all elements $e$ such that $M(e) \geq 1$.

We associate with each slice $\mathcal{G}_s = (\mathbb{D} \times \{y_1, y_2\}, \tau)$ of a globally fulfilling compass structure $\mathcal{G}$, the multiset $M$ defined by $M(F,G) = \big|\{x \in \mathbb{D} : \tau(x, y_1) = F,\ \tau(x, y_2) = G\}\big|$ for all $(F,G) \in (\mathsf{atoms}(\varphi) \uplus \{\varnothing\})^2$. We call this multiset *the profile of the slice* $\mathcal{G}_s$ and we denote it by $\mathsf{profile}(\mathcal{G}_s)$. Note that the projection $\mathsf{profile}(\mathcal{G}_s)|_1$ (resp., $\mathsf{profile}(\mathcal{G}_s)|_2$) onto the first (resp., second) component is a multiset that represents the number of occurrences of each atom along the lower (resp., upper) row of the slice $\mathcal{G}_s$. Definition 1 below captures a more general notion of profile that does not refer to a particular compass structure. We will then introduce trees labelled with profiles as abstractions of compass structures.

**Definition 1.** A *profile* is a multiset $M$ of pairs of (possibly dummy) atoms $(F,G) \in (\mathsf{atoms}(\varphi) \uplus \{\varnothing\})^2$ such that: (i) for all $(F,G) \in M$, if $F \neq \varnothing$, then $G \neq \varnothing$ and $F \uparrow G$; (ii) for all $(F,G), (F',G') \in M$, $\mathsf{req}_A(F) = \mathsf{req}_A(F')$ and $\mathsf{req}_A(G) = \mathsf{req}_A(G')$; (iii) $M$ contains infinitely many occurrences of pairs $(\varnothing, G)$ with $G \neq \varnothing$; (iv) $M$ contains *exactly one* occurrence of a pair $(F,G)$ with $F$ $\pi$-atom and *exactly one* occurrence of a pair $(\varnothing, H)$ with $H$ $\pi$-atom (for short, we denote the two pairs $(F,G)$ and $(\varnothing,H)$ by $M_\pi$ and $M^\pi$, respectively); (v) if $M_\pi = (F,G)$, then $\mathsf{req}_{\bar{A}}(F) = \bigcup_{(F',G') \in M} \mathsf{obs}(F')$; similarly, if $M^\pi = (\varnothing, H)$, then $\mathsf{req}_{\bar{A}}(H) = \bigcup_{(F',G') \in M} \mathsf{obs}(G')$.

**Definition 2.** A *profile tree* is an infinite finitely-branching tree $\mathcal{T} = (T, N, E)$, where $T$ is a decomposition of some dense order with endpoints, $N$ is a function mapping nodes of $T$ to profiles, and $E$ is a function mapping nodes of $T$ to multisets of tuples of atoms, such that:

- **(profile-match)** every node $s \in T$ has at least two children, say, $s_1, \dots, s_n$, with $n \geq 2$, and $E(s)$ is a multiset of $(n+1)$-tuples such that $E(s)|_{1,n+1} = N(s)$ and $E(s)|_{i,i+1} = N(s_i)$ for all $1 \leq i \leq n$;
- **(profile-finite-req)** for every node $s \in T$ and pair $(F_1, F_{n+1}) \in N(s)$, with $F_1 \neq \varnothing$, if $N(s)(F_1, F_{n+1}) < \infty$, then $E(s)$ contains *exactly* $N(s)(F_1, F_{n+1})$ occurrences of tuples $(F_1, \dots, F_{n+1})$ such that $\mathsf{req}_{\bar{B}}(F_1) = \bigcup_{2 \leq i \leq n+1} \mathsf{obs}(F_i) \cup \mathsf{req}_{\bar{B}}(F_{n+1})$ and $\mathsf{req}_B(F_{n+1}) = \bigcup_{1 \leq i \leq n} \mathsf{obs}(F_i) \cup \mathsf{req}_B(F_1)$;
- **(profile-infinite-req)** for every node $s \in T$ and pair $(F_1, F_{n+1}) \in N(s)$, with $F_1 \neq \varnothing$, if $N(s)(F_1, F_{n+1}) = \infty$, then $E(s)$ contains *at least one* occurrence of a tuple $(F_1, \dots, F_{n+1})$ such that $\mathsf{req}_{\bar{B}}(F_1) = \bigcup_{2 \leq i \leq n+1} \mathsf{obs}(F_i) \cup \mathsf{req}_{\bar{B}}(F_{n+1})$ and $\mathsf{req}_B(F_{n+1}) = \bigcup_{1 \leq i \leq n} \mathsf{obs}(F_i) \cup \mathsf{req}_B(F_1)$;
- **(profile-dummy)** for every node $s \in T$ and pair $(\varnothing, G) \in N(s)$, with $G \neq \varnothing$, $E(s)$ contains at least one occurrence of a $\pi$-*tuple*, i.e., a tuple with a $\pi$-atom, of the form $(F_1, \dots, F_{n+1})$, with $F_1 = \varnothing$ and $F_{n+1} = G$.

In addition, if the profile at the root $s_0$ of $\mathcal{T}$ contains only the pair $N^{\pi}(s_0) = (F, G)$, with $F$ $\pi$-atom and $\mathsf{req}_{\bar{B}}(G) = \varnothing$, and some pairs of the form $(\varnothing, H)$, with $H \neq \varnothing$ and $\mathsf{req}_{\bar{B}}(H) = \varnothing$, then $\mathcal{T}$ is said to be a *full* profile tree.

The first item of Definition 2 enforces the matching conditions between the pairs in the profile of a node and the pairs in the profiles of its children. The second item requires that all requests that appear in a pair $(F, G)$ of the profile of a node $s$ are either "locally fulfilled" by the observables of corresponding pairs in the profiles of the children or transferred to other nodes of the profile tree at the same level as $s$. This condition, however, concerns only those pairs $(F, G)$ that have finite multiplicity in the profile; for the remaining pairs, we enforce a similar, but weaker condition (third item of the definition). Finally, the fourth item requires that for each atom $G$, if the profile $N(s)$ contains the pair $(\varnothing, G)$, then at least one occurrence of this pair is "refined" in the multiset $E(s)$ by an occurrence of a tuple of the form $(\varnothing, \ldots, \varnothing, F, \ldots, G)$ that contains a $\pi$-atom $F$ (such a tuple is called for short $\pi$*-tuple*) and that ends with the atom $G$ (possibly $F = G$). We will see later that this condition is necessary for the fulfilment of the requests along the direction $\bar{A}$.

Below, we show that full profile trees are correct (though not yet finite) abstractions of globally fulfilling compass structures. We present this result with two statements showing, respectively, completeness and a weak form of soundness of profile trees. Note that the two-way correspondence is sufficient for witnessing satisfiability of $\mathsf{A\bar{A}B\bar{B}}$ formulas by means of profile trees.

**Proposition 2.** For every globally fulfilling compass structure $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$ over a dense order with endpoints $\mathbb{D}$, there is a full profile tree $\mathcal{T} = (T, N, E)$ such that $T$ is a decomposition of $\mathbb{D}$ and, for all nodes $s \in T$, $N(s) = \mathsf{profile}(\mathcal{G}_s)$. Conversely, for every full profile tree $\mathcal{T} = (T, N, E)$, with $T$ decomposition of some dense order with endpoints $\mathbb{D}$, there is a globally fulfilling compass structure $\mathcal{G} = (\mathbb{D}' \times \mathbb{D}', \tau)$, with $\mathbb{D}' \subseteq \mathbb{D}$ dense order with endpoints, such that $\mathsf{set}(\mathsf{profile}(\mathcal{G}_s)) = \mathsf{set}(N(s))$ for all $s \in T$.

Below, we show how to further restrict ourselves to a complete subset of full profile trees and derive *finite* representations of them. The general idea is to normalise profile trees so as to obtain structures that are sufficiently "regular" to be represented by finite trees. To this end, we introduce a finite variant of the notion of profile tree, called *finite profile tree*, that is obtained by enforcing the conditions of Definition 2 to internal nodes only (accordingly, since the multisets $E(s)$ that are associated with the leaves $s$ in a finite profile tree are not anymore relevant, one can assume that the function $E$ is undefined on the leaves). We also introduce a strengthening of the containment relation on multisets, denoted by $\sqsubseteq$ and defined as follows: $M \sqsubseteq N$ iff $\mathsf{set}(M) = \mathsf{set}(N)$ and $M(\bar{F}) \leq N(\bar{F})$ (resp., $M(\bar{F}) = N(\bar{F})$) for all tuples $\bar{F}$ (resp., $\pi$-tuples $\bar{F}$). The following definition captures precisely the set of profile trees we are interested in.

**Definition 3.** Let $\mathcal{T}$ be a finite or infinite profile tree. We say that $\mathcal{T}$ is *pseudo-regular* iff for all paths $\pi$, there are $s, s' \in \pi$, with $s$ proper ancestor of $s'$, such that $N(s) \sqsubseteq N(s')$ and $N(s)(\varnothing, G) = N(s')(\varnothing, G)$ for all atoms $G$.

In the following, we mainly work with profiles that appear at the roots of infinite profile trees (*feasible* profiles for short). We observe that the restriction of the partial order $\sqsubseteq$ to feasible profiles is a *well* partial order: indeed, the definition of profile tree implies that every $\pi$-tuple has multiplicity either 0 or 1 in any feasible profile, which in turn means that $\sqsubseteq$ is the conjunction of the well partial order $\subseteq$ and an equivalence of finite index. Hence, by a combination of Dickson's and König's lemmas, every infinite pseudo-regular tree has a finite prefix that is also pseudo-regular (a prefix of a tree is any restriction of it to an upward-closed set of nodes). A converse result also holds:

**Proposition 3.** For every finite pseudo-regular profile tree $\mathcal{T}$, there is an infinite profile tree $\mathcal{T}'$ that has the same profile as $\mathcal{T}$ at the root.

The crux of our semi-decision procedure for testing the satisfiability of $\mathsf{A\bar{A}B\bar{B}}$ formulas is to enumerate all atoms that appear in feasible profiles. Proposition 3 allows us to use finite pseudo-regular profile trees as witnesses of existence of some of these atoms. Unfortunately, this is not yet the end of the story, because not all profile trees are pseudo-regular and hence, a priori, there might exist atoms that appear only in infinite profile trees that are not pseudo-regular. The last piece of the puzzle amounts at showing that this is not the case and that we can indeed safely restrict ourselves to atoms appearing in pseudo-regular profile trees. We will prove this result by normalizing infinite profile trees via a series of operations that "inflate" the profiles as much as possible.

An important aspect that must be taken into account while inflating the profiles in a tree is that there are matching constraints to satisfy. As a matter of fact, these constraints induce dependencies between the multiplicities of pairs $(\varnothing, F)$ in the profile associated with a node $s$ and the multiplicities of corresponding pairs $(F, G)$ in the profile associated with the right sibling of $s$. As a consequence, there will be differences in the treatment of pairs of the form $(\varnothing, F)$ and pairs of the form $(F, G)$, with $F \neq \varnothing$. We take a brief interlude to give an example of the type of dependencies that can be enforced.

**Example 1.** Consider a formula $\varphi$ that contains, among other conjuncts, the subformula $[\mathsf{G}](a \rightarrow [\mathsf{B}]\neg a \wedge [\mathsf{\bar{B}}]\neg a)$. Figure 3 describes a slice of a compass structure that may satisfy $\varphi$, with some distinguished points annotated with observables and requests. The formula requires that all $a$-labelled points lie on distinct vertical axes; on the other hand, it allows arbitrarily many $a$-labelled points to be horizontally aligned. This is a representative example because, in general, forbidding multiple occurrences of an observable along the same horizontal line can be only done using the modal operator $[\mathsf{E}]$, which is not available in the logic. As concerns the multiplicities of the example profile, we observe that by inserting multiple $a$-labelled points along a single horizontal line and by accordingly modifying the upper part of the compass structure, one can get as many pairs of atoms $(F, G)$, where $\langle\mathsf{\bar{B}}\rangle a \in F$ and $[\mathsf{\bar{B}}]\neg a \in G$. On the other hand, to increase the number of pairs $(\varnothing, F)$, where $\langle\mathsf{\bar{B}}\rangle a \in F$, one has to introduce new horizontal lines ending with $\pi$-atoms $H$ such that $\langle\mathsf{\bar{A}}\rangle a \in$

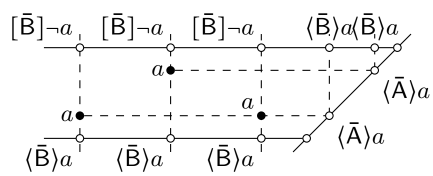




**Fig. 3.** Dependencies between multiplicities.

$H$: this is not always possible as other conjuncts of $\varphi$ may enforce bounds to the number of $\pi$-atoms $H$.

As shown by the above example, the simplest way one can inflate a profile, while preserving its feasibility, is by increasing the multiplicites associated with the pairs $(F, G)$, where $F \neq \emptyset$. We formalise this in the next lemma.

**Lemma 1.** If $N$ is a feasible profile and $N'$ is a profile such that $N \sqsubseteq N'$ and $N(\emptyset, G) = N'(\emptyset, G)$ for all atoms $G$, then $N'$ is feasible too. Moreover, a profile tree with root profile $N'$ can be obtained from a profile tree with root profile $N$ without modifying the underlying decomposition tree.

We describe a second inflation method, which depends on the previous one and can be used to further simplify the reasoning on the matching conditions of a profile tree $\mathcal{T} = (T, N, E)$. In particular, it shows that w.l.o.g. one can assume that the *finiteness* of the multiplicity of any tuple $(F_1, \ldots, F_{n+1})$ in a multiset $E(s)$ depends only on the multiplicity of the first component $F_1$ in $E(s)|_1$. This property is formalized below by the definition of "pointwise fair" profile tree, followed by a corresponding lemma that shows how to enforce the property.

**Definition 4.** A multiset $E$ of $(n+1)$-tuples is *fair* if for all $(n+1)$-tuples $(F_1, \ldots, F_{n+1}) \in E$, with $F_1 \neq \emptyset$, $E|_1(F_1) = \infty$ implies $E(F_1, \ldots, F_{n+1}) = \infty$. A profile tree $\mathcal{T} = (T, N, E)$ is *pointwise fair* if all multisets $E(s)$ are fair.

**Lemma 2.** For every feasible profile $N$, there is an infinite pointwise fair profile tree that has root profile $N' \sqsupseteq N$. Moreover, one can assume that, for all pairs of atoms $(F, G)$, if $N|_1(F) < \infty$, then $N(F, G) = N'(F, G)$.

A third inflation method makes use of the fact that the partial order $\sqsubseteq$ restricted to the set of feasible profiles is $\omega$-complete.

**Lemma 3.** Every sequence of feasible profiles $N_0 \sqsubseteq N_1 \sqsubseteq \ldots$ has a supremum $\sup_i N_i$, defined by $(\sup_i N_i)(F, G) = \sup_{i \in \mathbb{N}} \big(N_i(F, G)\big)$ for all atoms $F, G$, that is a feasible profile.

We have described three ways of increasing the multiplicities of profiles at the roots of profile trees. In general, these techniques are not applicable to nodes that are strictly below the root. This is why we introduce a new partial order $\trianglelefteq$, incomparable with $\sqsubseteq$, that is defined only over feasible profiles $N, N'$ as follows:

$$N \trianglelefteq N' \qquad \text{iff} \qquad \begin{cases} N \subseteq N' \\ \mathsf{set}(N|_2) = \mathsf{set}(N'|_2) \\ N(F, G) = N'(F, G) & \text{for all atoms } F, G \neq \emptyset. \end{cases}$$

We observe that from any infinite $\trianglelefteq$-chain of feasible profiles, one can extract an infinite sub-sequence that is also a $\sqsubseteq$-chain. Thus, an immediate consequence of Lemma 3 is that every $\trianglelefteq$-chain has an *upper bound*. In its turn, the existence of upper bounds on $\trianglelefteq$-chains implies the existence of feasible profiles that are *maximal* with respect to $\trianglelefteq$ (this can be seen as a consequence of Zorn's Lemma):

**Corollary 1.** For all feasible profiles $N$, there is a $\trianglelefteq$-maximal profile $N' \trianglerighteq N$.

Based on existence of $\trianglelefteq$-maximal profiles, we say that a profile tree is *pointwise $\trianglelefteq$-maximal* if all its profiles are $\trianglelefteq$-maximal. Below, we show that all atoms of feasible profiles appear at the roots of some pseudo-regular profile trees.

**Proposition 4.** For every infinite pointwise fair profile tree with root profile $N$, there is an infinite pointwise fair and pointwise $\trianglelefteq$-maximal profile tree with root profile $N' \trianglerighteq N$.

**Proposition 5.** Every infinite pointwise fair and pointwise $\trianglelefteq$-maximal profile tree is pseudo-regular.

Wrapping up, we can devise a semi-decision procedure that tests the satisfiability of a formula $\varphi$ over $\mathbb{Q}$. The procedure works as follows. It first transforms $\varphi$ into an equi-satisfiable formula $\varphi_{]\,[}$ interpreted over a dense order with endpoints $\mathbb{D}$. Then, the procedure enumerates all finite full pseudo-regular trees, until a tree is found that contains the formula $\langle \mathsf{G} \rangle \varphi_{]\,[}$ as an observable of one of its atoms. The above semi-decision procedure is correct, namely, it terminates successfully iff the input formula $\varphi$ is satisfiable over $\mathbb{Q}$. Indeed, if $\varphi$ is satisfiable over $\mathbb{Q}$, then $\varphi_{]\,[}$ is satisfiable over a dense order with endpoints $\mathbb{D}$, and hence there is a globally fulfilling compass structure $\mathcal{G}$ that contains $\langle \mathsf{G} \rangle \varphi_{]\,[}$ as an observable of all its atoms. By Propositions 4 and 5, there is also an infinite, pseudo-regular full profile tree $\mathcal{T}$ that witnesses $\langle \mathsf{G} \rangle \varphi_{]\,[}$ at the root profile. By the remarks that follow Definition 3, there is also a prefix of $\mathcal{T}$ that is a finite pseudo-regular full profile tree, and eventually this tree must be discovered by the procedure. Conversely, if the procedure terminates with a finite pseudo-regular full profile tree witnessing $\langle \mathsf{G} \rangle \varphi_{]\,[}$, then by Proposition 3 there is an infinite full profile tree $\mathcal{T}$, and hence a compass structure $\mathcal{G}$, that witness the satisfiability of $\langle \mathsf{G} \rangle \varphi_{]\,[}$ over $\mathbb{D}$. One can then conclude that $\varphi$ is satisfiable over $\mathbb{Q}$.

A full decision procedure that solves the satisfiability problem for $\mathsf{A\bar{A}B\bar{B}}$ over $\mathbb{Q}$ can simply run in parallel the two semi-decision procedures that we described for unsatisfiability and satisfiability of $\mathsf{A\bar{A}B\bar{B}}$ formulas.

As for the satisfiability problem over the class of all interval structures, one can simply observe the following. The logic $\mathsf{A\bar{A}B\bar{B}}$, as any other HS fragment, can be viewed as a fragment of first-order logic that uses binary relations to express properties of pairs of elements of the underlying temporal domain. The relation $<$ of the temporal domain can be easily constrained by a first-order formula so as to define a linear order, and Allen's relations can be expressed in first-order logic in term of $<$. From Löwenheim-Skolem theorem, it follows that every interval structure can be assumed to contain only *countably* many intervals. Moreover, since every countable linear order can be embedded inside $\mathbb{Q}$, satisfiability of formulas of a given HS fragment over the class of all linear orders can be reduced to their satisfiability over $\mathbb{Q}$, provided that the fragment is powerful enough to express such an embedding. This is the case with $\mathsf{A\bar{A}B\bar{B}}$: it suffices to introduce a distinguished proposition letter #, to constrain all #-labelled intervals to be singletons ($[\mathsf{G}](\# \to \pi)$), and to relativize all modalities to intervals with endpoints labelled by # (intervals that satisfy $\langle \mathsf{B} \rangle \# \wedge \langle \mathsf{A} \rangle \#$). We conclude by establishing the precise complexity of the satisfiability problem.

**Theorem 2.** The satisfiability problem for $\mathsf{A\bar{A}B\bar{B}}$ interpreted over $\mathbb{Q}$, as well as over the class of all linear orders, is decidable, but not primitive recursive.

## 5 Conclusions

In this paper we close the open questions concerning the satisfiability problem for the interval temporal logic $\mathsf{A\bar{A}B\bar{B}}$. First, we generalized the undecidability result from [11] to $\mathbb{R}$ and to the class of all Dedekind-complete linear orders, and then we proved that it is decidable in two interesting cases: $\mathbb{Q}$ and the class of all interval structures. To decide satisfiability of $\mathsf{A\bar{A}B\bar{B}}$ formulas over $\mathbb{Q}$ we used a combination of techniques from [4] (tree-shaped decomposition of models) and [11] (encoding of models by systems with counters), plus new key ingredients (separation into two semi-decision procedures, Konig's lemma). As concerns the second result, the decidability of $\mathsf{A\bar{A}B\bar{B}}$ over the class of all interval structures follows from the decidability over $\mathbb{Q}$ and from Löwenheim-Skolem theorem, which allows us to assume, without loss of generality, that the interval structures are countable and hence embeddable inside $\mathbb{Q}$. The fact that $\mathsf{A\bar{A}B\bar{B}}$ is powerful enough to express the embedding of a countable order inside $\mathbb{Q}$ completes the reduction. It is worth pointing out that the same technique cannot be applied to all HS fragments; for instance, the satisfiability problem for the temporal logic of sub-/super-intervals $\mathsf{D\bar{D}}$ is known to be decidable over $\mathbb{Q}$ [2, 10], but it is open for the class of all interval structures.

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# A Appendix

## A.1 Undecidability over Dedekind-complete orders embedding $\mathbb{N}$

**Theorem 1.** *The satisfiability problem for* $\mathsf{A\bar{A}B}$ *interpreted over* $\mathbb{N}$, $\mathbb{R}$, *and the class of all Dedekind-complete linear orders is undecidable.*

*Proof.* The proof is divided into two parts. In the first part, we reduce a variant of an undecidable reachability problem for lossy counter machines to the satisfiability problem for $\mathsf{A\bar{A}B}$ interpreted over the temporal domain $\mathbb{N}$. In the second part we show how to lift such an undecidability result to the temporal domain $\mathbb{R}$ and to the class of all Dedekind-complete linear orders. The proof for the first part was essentially given already in [11] (we only made some small improvements to the proof); we report it to make the proof self-contained.

As a preliminary step, we recall the precise definition of a *lossy (Minsky) counter machine*. This is a triple of the form $\mathcal{M} = (Q, k, \delta)$, where $Q$ is a finite set of control states, $k$ is the number of counters, whose values range over $\mathbb{N}$, and $\delta$ is a function that maps each state $q \in Q$ to a transition rule having one of the following forms:

- $\texttt{inc}(i)$ $\texttt{and goto}(q')$, where $i \in \{1, ..., k\}$ is a counter and $q' \in Q$ is a state. The meaning of this transition is that, whenever $\mathcal{M}$ is in state $q$, then $\mathcal{M}$ must increment the value of counter $i$ and switch to state $q'$.
- $\texttt{if}$ $i = 0$ $\texttt{then goto}(q')$ $\texttt{else dec}(i)$ $\texttt{and goto}(q'')$, where $i \in \{1, ..., k\}$ is a counter and $q', q'' \in Q$ are states. The meaning of this transition is that, whenever $\mathcal{M}$ is in state $q$ and the value of the counter $i$ is 0 (resp., greater than 0), then $\mathcal{M}$ must switch to state $q'$ (resp., it must decrement the value of the counter $i$ and switch to state $q''$).

In addition, from each configuration $(q, \bar{z}) \in Q \times \mathbb{N}^k$, $\mathcal{M}$ can non-deterministically activate an internal (lossy) transition and move to a configuration $(q, \bar{z}')$, with $\bar{z}' \leq \bar{z}$ (the relation $\leq$ is defined componentwise on the values of the counters). A computation of $\mathcal{M}$ is any sequence of configurations that respects the semantics of the transition relation.

The variant of the reachability problem we want to reduce from is called *structural termination* and consists of deciding, given a lossy counter machine $\mathcal{M} = (Q, k, \delta)$ and a pair of control states $q_{\mathsf{init}}$ and $q_{\mathsf{halt}}$, whether every computation of $\mathcal{M}$ that stars at state $q_{\mathsf{init}}$, with any arbitrary assignment for the counters, eventually reaches the state $q_{\mathsf{halt}}$, again with an arbitrary assignment for the counters. We know from the results in [9] that structural termination is undecidable.

To explain the reduction from structural termination of a lossy counter machine $\mathcal{M}$ to satisfiability of an $\mathsf{A\bar{A}B}$ formula, consider a generic *infinite* computation of $\mathcal{M}$ of the form

$$(q_1, \bar{z}_1) \quad (q_2, \bar{z}_2) \quad \ldots$$

that starts at the initial state $q_1 = q_{\mathsf{init}}$ and that avoids the halting state $q_{\mathsf{halt}}$, that is, $q_t \neq q_{\mathsf{halt}}$ for all $t \geq 1$. We encode the above computation into a suitable interval structure $\mathcal{I} = (\mathbb{I}_{\mathbb{N}}, \sigma, A, \bar{A}, B)$.

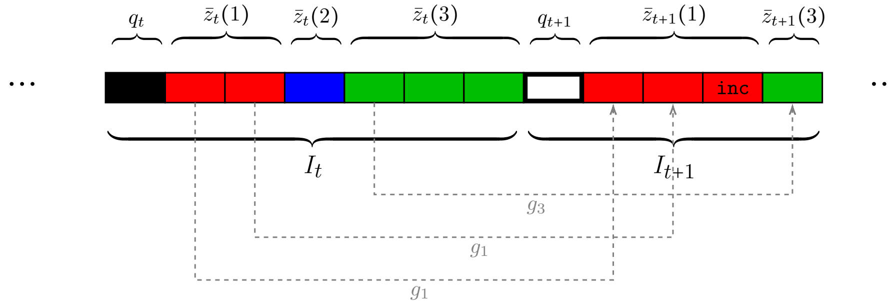


**Fig. 4.** Encoding of part of a computation of a lossy counter machine.

First, we divide the temporal domain $\mathbb{N}$ into an infinite sequence of intervals $I_1 = [x_1, x_2]$, $I_2 = [x_2, x_3]$, ..., called *blocks*, where $1 = x_1 < x_2 < \ldots$ and $x_{t+1} - x_t = 1 + \sum_{1 \le i \le k} \bar{z}_t(i)$ for all $t \geq 1$. We also introduce an additional dummy block $I_0 = [0, 1]$ to correctly move between the various blocks via the modal operators $\langle \mathsf{A} \rangle$ and $\langle \bar{\mathsf{A}} \rangle$. We observe that all pairs of consecutive blocks $I_t, I_{t+1}$ satisfy the "meet" relation $A$, namely, $I_t \; A \; I_{t+1}$ for all all $t \geq 0$.

Next, we introduce $|Q| + k$ proposition letters that will label the *unit-length intervals* $[x, x+1]$ that are contained in each block $I_t$: the first $|Q|$ proposition letters will be identified with the control states $q \in Q$ of $\mathcal{M}$, while the last $k$ proposition letters, denoted $c_1, \ldots, c_k$, will represent the names of the $k$ counters of $\mathcal{M}$. Let $C = \{c_1, \ldots, c_k\}$. The labelling function $\sigma$ associate a unique proposition letter in $Q \cup C$ with each unit-length sub-interval of each block $I_t$, with $t \geq 1$, as follows:

1. the interval $[x_t, x_t + 1]$ is labelled by the control state $q_t$;
2. for every $1 \leq i \leq k$, the number of $c_i$-labelled intervals of the form $[x, x+1]$, with $x_t < x < x_{t+1}$, coincides with the value $\bar{z}_t(i)$ of the counter $i$;
3. all other intervals do not carry any letter from $Q \cup C$

(note that there exist different encodings of the same computation of $\mathcal{M}$).

As an example, Figure 4 represents part of an encoding of a computation for a lossy counter machine $\mathcal{M}$ with two control states, whose occurrences are represented by black-colored and white-colored intervals, and three counters, whose values are represented by the numbers of occurrences of intervals colored, respectively, by red, blue, and green (the meaning of the dashed arrows is explained below).

The next ingredient of the reduction consists of defining the validity of the encoding by means of a suitable $\mathsf{A\bar{A}B}$ formula $\varphi^{\mathcal{M}}$. In the following, we will prove that $\varphi^{\mathcal{M}}$ is satisfiable over the temporal domain $\mathbb{N}$ if and only if $\mathcal{M}$ admits an infinite computation starting at state $q_{\mathsf{init}}$ and avoiding the halting state $q_{\mathsf{halt}}$, that is, if and only if $\mathcal{M}$ is a negative instance of the structural termination problem.

We are specifically interested in enforcing inequalities between counters of the form $\bar{z}_{t+1}(i) \leq \bar{z}_t(i) + h$, with $h \in \{-1, 0, 1\}$. We first explain how this can be done for the case $h = 0$. By definition, enforcing a constraint of the form $\bar{z}_{t+1}(i) \leq \bar{z}_t(i)$ is equivalent to enforcing the existence of a *surjective partial function* $g_i$ from the set of $c_i$-labelled sub-intervals of $I_t$ to the set of $c_i$-labelled sub-intervals of $I_{t+1}$. As an example, the dashed arrow labelled by $g_3$ in Figure 4 represents one instance of a surjective partial function encoding a constraint of the form $\bar{z}_{t+1}(3) \leq \bar{z}_t(3)$. In its turn, every partial function $g_i$ can be represented by a set of intervals of the form $[x, g_i(x)]$, with $x_t < x < x_{t+1} < g_i(x) < x_{t+2}$ and $\sigma([x, x+1]) = \sigma([g_i(x), g_i(x)+1]) = c_i$, which can then be labelled by a fresh proposition letter $g_i$. The relevant properties of these $g_i$-labelled intervals are translated to a suitable formula $\varphi_i^{\leq}$ evaluated over the block $I_t$. Precisely, we define

$$\begin{aligned}\varphi_i^{\leq} \;=\; & [\mathsf{B}][\mathsf{A}]\big(g_i \rightarrow (\varphi_Q^{\exists!} \wedge [\mathsf{B}]\neg g_i \wedge \langle\mathsf{B}\rangle c_i \wedge \langle\mathsf{A}\rangle c_i)\big) \;\wedge \\ & [\mathsf{B}][\mathsf{A}]\big((\varphi_Q^{\exists!} \wedge \langle\mathsf{A}\rangle c_i) \rightarrow \langle\mathsf{A}\rangle\langle\bar{\mathsf{A}}\rangle g_i\big),\end{aligned}$$

where $\varphi_Q^{\exists!} = \langle\mathsf{B}\rangle\langle\mathsf{A}\rangle \bigvee_{q\in Q} q \wedge [\mathsf{B}]\big([\mathsf{A}]\bigvee_{q\in Q} q \rightarrow [\mathsf{B}][\mathsf{A}]\bigwedge_{q\in Q}\neg q\big)$. Intuitively, $\varphi_Q^{\exists!}$ holds at some interval $J$ iff $J$ contains *exactly one* unit-length sub-interval labelled by some proposition letter in $Q$. The first line of the formula $\varphi_i^{\leq}$ above enforces the condition that the set of $g_i$-labelled intervals that start inside $I_t$ represents a partial function from the $c_i$-labelled sub-intervals of $I_t$ to the $c_i$-labelled sub-intervals of $I_{t+1}$. The second line of $\varphi_i^{\leq}$ guarantees that such a partial function is surjective.

In a similar way, one can enforce a constraint of the form $\bar{z}_{t+1}(i) \leq \bar{z}_t(i) - 1$ (resp., $\bar{z}_{t+1}(i) \leq \bar{z}_t(i) + 1$) by means of a formula $\varphi_{\mathtt{dec}(i)}^{\leq}$ (resp., $\varphi_{\mathtt{inc}(i)}^{\leq}$). This is done by excluding from the domain (resp., from the range) of the surjective partial function $g_i$ exactly one $c_i$-labelled sub-interval of $I_t$ (resp., $I_{t+1}$), which is thus distinguished by means of an additional proposition letter $\mathsf{dec}$ (resp., $\mathsf{inc}$). Precisely, we let

$$\begin{aligned}\varphi_{\mathtt{dec}(i)}^{\leq} = \varphi_{\{\mathsf{dec}\}}^{\exists!} \;\wedge & \\ & [\mathsf{B}][\mathsf{A}]\big(\, g_i \;\rightarrow\; (\varphi_Q^{\exists!} \wedge [\mathsf{B}]\neg g_i \wedge \langle\mathsf{B}\rangle(c_i \wedge \neg\mathsf{dec}) \wedge \langle\mathsf{A}\rangle c_i)\,\big) \;\wedge \\ & [\mathsf{B}][\mathsf{A}]\big(\,(\varphi_Q^{\exists!} \wedge \langle\mathsf{A}\rangle c_i) \;\rightarrow\; \langle\mathsf{A}\rangle\langle\bar{\mathsf{A}}\rangle g_i\,\big)\end{aligned}$$

$$\begin{aligned}\varphi_{\mathtt{inc}(i)}^{\leq} = \langle\mathsf{A}\rangle\big(\,[\mathsf{B}](\neg\pi \;\rightarrow\; [\mathsf{A}]\textstyle\bigwedge_{q\in Q}\neg q) \;\wedge\; \langle\mathsf{A}\rangle(\bigvee_{q\in Q} q \wedge \varphi_{\{\mathsf{inc}\}}^{\exists!})\,\big) \;\wedge & \\ & [\mathsf{B}][\mathsf{A}]\big(\, g_i \;\rightarrow\; (\varphi_Q^{\exists!} \wedge [\mathsf{B}]\neg g_i \wedge \langle\mathsf{B}\rangle c_i \wedge \langle\mathsf{A}\rangle(c_i \wedge \neg\mathsf{inc}))\,\big) \;\wedge \\ & [\mathsf{B}][\mathsf{A}]\big(\,(\varphi_Q^{\exists!} \wedge \langle\mathsf{A}\rangle(c_i \wedge \neg\mathsf{inc})) \;\rightarrow\; \langle\mathsf{A}\rangle\langle\bar{\mathsf{A}}\rangle g_i\,\big)\end{aligned}$$

where $\varphi_X^{\exists!}$ is defined as before for a generic set $X$ of proposition letters.

It now remains to translate each transition rule $\delta(q)$ of $\mathcal{M}$ into a corresponding formula $\varphi_q^{\delta}$ taking into account the various possible cases. This can be done as follows:

1. if $\delta(q)$ is a transition rule of the form $\mathtt{inc}(i)$ $\mathtt{and}$ $\mathtt{goto}(q')$, then we put $\varphi_q^{\delta} = \langle\mathsf{A}\rangle q' \wedge \varphi_{\mathtt{inc}(i)}^{\leq} \wedge \bigwedge_{j\neq i}\varphi_j^{\leq}$;
2. if $\delta(q)$ is a transition rule of the form $\mathtt{if}$ $i = 0$ $\mathtt{then}$ $\mathtt{goto}(q')$ $\mathtt{else}$ $\mathtt{dec}(i)$ $\mathtt{and}$ $\mathtt{goto}(q'')$, then we put $\varphi_q^{\delta} = \big([\mathsf{B}][\mathsf{A}]\neg c_i \rightarrow \varphi_{q,0}^{\delta}\big) \wedge \big(\langle\mathsf{B}\rangle\langle\mathsf{A}\rangle c_i \rightarrow \varphi_{q,1}^{\delta}\big)$, where $\varphi_{q,0}^{\delta} = \langle\mathsf{A}\rangle q' \wedge \bigwedge_{1\leq i\leq k}\varphi_i^{\leq}$ and $\varphi_{q,1}^{\delta} = \langle\mathsf{A}\rangle q'' \wedge \varphi_{\mathtt{dec}(i)}^{\leq} \wedge \bigwedge_{j\neq i}\varphi_j^{\leq}$.

The set of all infinite computations of $\mathcal{M}$ that start at $q_{\mathsf{init}}$ and avoid $q_{\mathsf{halt}}$ is captured by the following $\mathsf{A\bar{A}B}$ formula:

$$\begin{aligned}\varphi^{\mathcal{M}} = [\mathsf{G}]\Big((\langle\mathsf{A}\rangle\neg\pi \wedge \langle\bar{\mathsf{A}}\rangle\neg\pi \wedge \mathsf{unit}) \;\rightarrow\; \big(\bigvee_{a\in Q\cup C} a \;\wedge \bigwedge_{a\neq b\in Q\cup C}\neg(a\wedge b)\big)\Big) \wedge \\ [\mathsf{G}]\Big(((\mathsf{unit} \wedge ([\mathsf{A}]\pi \vee [\bar{\mathsf{A}}]\pi)) \vee \neg\mathsf{unit}) \;\rightarrow \bigwedge_{a\in Q\cup C\cup\{\mathsf{inc},\mathsf{dec}\}}\neg a\Big) \wedge \\ [\mathsf{G}]\bigwedge_{q\in Q}\Big(\langle\mathsf{B}\rangle q \;\rightarrow\; \langle\bar{\mathsf{A}}\rangle\langle\mathsf{A}\rangle\varphi^{\delta}_{q}\Big) \;\wedge\; [\mathsf{G}]\neg q_{\mathsf{halt}} \;\wedge\; \langle\mathsf{G}\rangle q_{\mathsf{init}}.\end{aligned}$$

It can be easily checked that $\varphi^{\mathcal{M}}$ is satisfiable over $\mathbb{N}$ if and only if there exists an initial configuration of the form $(q_{\mathsf{init}}, \bar{z})$, with $\bar{z} \in \mathbb{N}^k$, from which the lossy counter machine $\mathcal{M}$ never halts.

We show now that the above undecidability result can be transferred to $\mathsf{A\bar{A}B}$ formulas interpreted over the order $\mathbb{R}$ of the reals, or even over the class of all Dedekind-complete linear orders. To this end, we introduce a fresh proposition letter $\#$, which labels singleton intervals only. To lift the undecidability result from $\mathbb{N}$ to $\mathbb{R}$ it suffices to enforce that *the set of all $\#$-labelled singleton intervals, with the ordering inherited from $\mathbb{R}$, is isomorphic to* $\mathbb{N}$. This can be done by means of the following $\mathsf{A\bar{A}B}$ formula $\varphi_{\#}$:

$$\begin{aligned}\varphi_{\#} = [\mathsf{G}]\big(\# \rightarrow \pi\big) \;\wedge\; \langle\mathsf{G}\rangle\big(\# \wedge [\bar{\mathsf{A}}](\neg\pi \rightarrow (\neg\# \wedge [\bar{\mathsf{A}}]\neg\#))\big) \;\wedge \\ [\mathsf{G}]\big(\# \rightarrow \langle\mathsf{A}\rangle(\neg\pi \wedge \langle\mathsf{A}\rangle\# \wedge [\mathsf{B}](\neg\pi \rightarrow [\mathsf{A}]\neg\#))\big) \;\wedge \\ [\mathsf{G}]\big((\pi \wedge [\bar{\mathsf{A}}](\neg\pi \rightarrow \langle\mathsf{B}\rangle\langle\mathsf{A}\rangle\#)) \rightarrow [\mathsf{A}][\mathsf{A}]\neg\#\big).\end{aligned}$$

The three components (lines) of $\varphi_{\#}$ respectively enforce the following conditions: (i) all $\#$-labelled intervals are singletons and there exists one such interval that has no other $\#$-labelled intervals strictly to the left, (ii) every $\#$-labelled interval is followed strictly to the right by another $\#$-labelled interval and in between there are no other $\#$-labelled intervals (existence of immediate $\#$-successors), (iii) for every point $y$ in the temporal domain $\mathbb{R}$, if there are occurrences of $\#$ arbitrarily close to $y$ from the left (namely, if $y$ is a right accumulation point of $\#$), then there is no occurrence of $\#$ at $y$ or to its right. In particular, since $\mathbb{R}$, where $\varphi_{\#}$ is interpreted, is Dedekind complete, the last condition implies that every $\#$-labelled interval sees only finitely many occurrences of $\#$ to the left (otherwise, there would exist an accumulation point $y$ violating the condition), and thus the set of $\#$-labelled singleton intervals is isomorphic to $\mathbb{N}$.

Now, in order to correctly encode the set of relevant computations of a lossy counter machine $\mathcal{M}$ into an interval structure $\mathcal{I} = (\mathbb{I}_{\mathbb{R}}, A, \bar{A}, B, \sigma)$, it suffices to pair $\varphi_{\#}$ with the formula obtained from $\varphi^{\mathcal{M}}$ by constraining each modality to range only over the intervals that satisfy $\langle\mathsf{B}\rangle\# \wedge \langle\mathsf{A}\rangle\#$, that is, those intervals with endpoints labelled by $\#$ (a correspondence between these intervals and the intervals over $\mathbb{N}$ can be naturally established). This shows that the satisfiability problem for $\mathsf{A\bar{A}B}$ interpreted over $\mathbb{R}$ is undecidable.

Finally, we extend the above undecidability result to the class of all Dedekind-complete linear orders. To this end, it suffices to add a fourth conjunct to the

$\tilde{\varphi} \leftarrow \langle \mathsf{G} \rangle \varphi$
$\mathcal{G}_0 \leftarrow$ **empty compass** $\tilde{\varphi}$**-structure**
$T \leftarrow$ **singleton tree with** $\mathcal{G}_0$ **as root**
$Q \leftarrow$ **empty queue**
$z' \leftarrow$ **new element**
***AddRefinements*** $(T, Q, \mathcal{G}_0, z', (z', z'), \tilde{\varphi})$

**while** $Q \neq \varnothing$
- $\mathsf{obj} \leftarrow$ **dequeue**$(Q)$
- **for all leaves** $\mathcal{G}$ **in** $T$
  - **let** $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$
  - **if** $\mathsf{obj} = (z, \mathsf{min})$ **and** $z = \min(\mathbb{D})$ **then**
    - **let** $z'$ **be a new element with** $z' < z$
    - ***AddRefinements*** $(T, Q, \mathcal{G}, z', (z', z'), \mathsf{true})$
  - **if** $\mathsf{obj} = (z, \mathsf{max})$ **and** $z = \max(\mathbb{D})$ **then**
    - **let** $z'$ **be a new element with** $z' > z$
    - ***AddRefinements*** $(T, Q, \mathcal{G}, z', (z', z'), \mathsf{true})$
  - **if** $\mathsf{obj} = (z_1, z_2)$ **and** $z_1, z_2$ **are consecutive in** $\mathbb{D}$ **then**
    - **let** $z'$ **be a new element with** $z_1 < z' < z_2$
    - ***AddRefinements*** $(T, Q, \mathcal{G}, z', (z', z'), \mathsf{true})$
  - **if** $\mathsf{obj} = ((x, y), R, \psi)$ **and** $\forall x', y' \in \mathbb{D}.$ $(x, y)\ R\ (x', y') \rightarrow \psi \notin \mathsf{obs}(\tau(x', y'))$ **then**
    - **let** $\mathbb{D} = \{z_1 < \ldots < z_n\}$
    - **for all** $i = 0 \ldots n$
      - **let** $z'$ **be a new element with** $z_1 < \ldots < z_i < z' < z_{i+1} < \ldots < z_n$
      - $\mathbb{D}' \leftarrow \mathbb{D} \cup \{z'\}$
      - **for all** $(x', y') \in (\mathbb{D}' \times \mathbb{D}') \smallsetminus (\mathbb{D} \times \mathbb{D})$ **with** $(x, y)\ R\ (x', y')$
        - ***AddRefinements*** $(T, Q, \mathcal{G}, z', (x', y'), \psi)$

**function** ***AddRefinements*** $(T, Q, \mathcal{G}, z', (x, y), \psi)$
- **let** $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$ **with** $\mathbb{D} = \{z_1 < \ldots < z_n\}$
- **for all atoms** $F$**,** $H_1, \ldots, H_n$**,** $V_1, \ldots, V_n$
  - $\mathbb{D}' \leftarrow \mathbb{D} \cup \{z'\}$
  - $\tau' \leftarrow \tau \ \cup\ \{(z', z') \mapsto F\}$ $\cup\ \{(z_i, z') \mapsto H_i : 1 \le i \le n\}$ $\cup\ \{(z', z_i) \mapsto V_i : 1 \le i \le n\}$
  - $\mathcal{G}' \leftarrow (\mathbb{D}' \times \mathbb{D}', \tau')$
  - **if** $\tau'(x, y) \vDash \psi$ **and** $\mathcal{G}'$ **is consistent then**
    - **insert** $\mathcal{G}'$ **as a new child of** $\mathcal{G}$ **in** $T$
    - **if** $z' = \min(\mathbb{D}')$ **then**
      - **enqueue**$(Q, (z', \mathsf{min}))$
    - **else**
      - $z \leftarrow$ **predecessor of** $z'$ **in** $\mathbb{D}'$
      - **enqueue**$(Q, (z, z'))$
    - **if** $z' = \max(\mathbb{D}')$ **then**
      - **enqueue**$(Q, (z', \mathsf{max}))$
    - **else**
      - $z \leftarrow$ **successor of** $z'$ **in** $\mathbb{D}'$
      - **enqueue**$(Q, (z', z))$
    - **for all** $(x', y') \in (\mathbb{D}' \times \mathbb{D}') \smallsetminus (\mathbb{D} \times \mathbb{D})$, $R' \in \{A, \bar{A}, B, \bar{B}\}$, $\psi' \in \mathsf{req}_{R'}(\tau'(x', y'))$
      - **enqueue**$(Q, ((x', y'), R', \psi'))$

**Fig. 5.** A semi-decision procedure that checks unsatisfiability of an $\mathsf{A\bar{A}B\bar{B}}$ formula $\varphi$.

above formula that selects a (Dedekind-complete) dense order, e.g.,

$$\varphi_{\mathsf{dense}} \ = \ [\mathsf{G}]\big(\neg\pi \rightarrow \langle \mathsf{B} \rangle \neg\pi\big).$$

□

### A.2 Enumerating unsatisfiable formulas over $\mathbb{Q}$

Here we provide a fairly simple semi-decision procedure that receives a formula $\varphi$ of $\mathsf{A\bar{A}B\bar{B}}$ as input and terminates (successfully) if and only if $\varphi$ is *unsatisfiable* over any interval structure with $\mathbb{Q}$ as temporal domain. The procedure is outlined in Figure 5. It starts by defining $\tilde{\varphi}$ as $\langle \mathsf{G} \rangle \varphi$ and by constructing the finite collection of all (consistent, but possibly not globally fulfilling) compass $\tilde{\varphi}$-structures that contain exactly one point and that feature the formula $\tilde{\varphi}$ as an observable. Clearly, every model of $\varphi$ over the rationals, if it exists, embeds at least one of these compass structures (the embedding relation is defined here as an isomorphism between the smaller structure and the restriction of the larger structure to a suitable subset of its temporal domain).

Next, the procedure repeatedly refines the compass structures in the collection by both forcing their temporal domains to be dense and fulfilling all requests of all their points. More precisely, a refinement step consists of adding a new element $z'$ to the domain $\mathbb{D}$ of a compass structure and guessing, in a way consistent with the previous choices, the ordering of $z'$ with respect to the elements in $\mathbb{D}$ and the atoms for the emerging points, namely, the points of the form $(x, z')$ and $(z', y)$, with $x, y \in \mathbb{D} \cup \{z'\}$ and $x \leq z' \leq y$.

The generated compass structures are stored as nodes of a tree $T$, whose edges represent the refinement relation over compass structures. In addition, the procedure adopts a refinement strategy which guarantees, in the limit, the following conditions for all compass structures $\mathcal{G}$ in the tree-shaped collection $T$:

1. along every infinite refinement path departing from $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$, the minimal (resp., maximal) element $z$ of the domain $\mathbb{D}$ is eventually replaced, in the role of minimal (resp., maximal) element, by a new minimal (resp., maximal) element $z' < z$ (resp., $z' > z$);
2. along every infinite refinement path departing from $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$, every two consecutive elements $z_1, z_2$ in the domain $\mathbb{D}$ are eventually separated by a new element $z'$, with $z_1 < z' < z_2$;
3. along every infinite refinement path departing from $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$, every request $\psi \in \mathsf{req}_R(\tau(x, y))$, for every point $(x, y)$ of $\mathcal{G}$ and every direction $R \in \{A, \bar{A}, B, \bar{B}\}$, is eventually fulfilled by the observables of another point.

To enforce the above properties, the procedure chooses in a *fair* way one among the following objects: (i) an element $z$ from the domain of some compass structure of $T$, (ii) a pair of elements $z_1 < z_2$ from the domain of some compass structure of $T$, (iii) a point $(x, y)$ and a request $\psi$ of it, for some direction $R \in \{A, \bar{A}, B, \bar{B}\}$, in some compass structure of $T$. Next, the procedure scans all compass structures that appear *at the leaves* of $T$, trying to satisfy finitary variants of conditions 1, 2, and 3 above. Specifically, if the selected object is the minimal (resp., maximal) element $z$ of the domain $\mathbb{D}$ of a leaf $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$ of $T$, then the procedure appends to $\mathcal{G}$ as many children as there are (non-isomorphic) refinements $\mathcal{G}' = (\mathbb{D}' \times \mathbb{D}', \tau')$ of $\mathcal{G}$ such that $z \neq \min(\mathbb{D}')$ (resp., $z \neq \max(\mathbb{D}')$) – in this way the element $z$ is replaced, in the role of minimal (resp., maximal) element, by a new element $z' \in \mathbb{D}' \setminus \mathbb{D}$. Similarly, if the selected object is a pair of elements $z_1 < z_2$ that are consecutive in the domain $\mathbb{D}$ of a leaf $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$ of $T$, then the procedure appends to $\mathcal{G}$ as many children as there are (non-isomorphic) refinements $\mathcal{G}' = (\mathbb{D}' \times \mathbb{D}', \tau')$ such that $z_1 < z' < z_2$ for some $z' \in \mathbb{D}' \setminus \mathbb{D}$. Finally, if the selected object consists of a point $(x, y)$ and a request $\psi \in \mathsf{req}_R(\tau(x, y))$ that is not fulfilled in a leaf $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$ of $T$, then the procedure appends to $\mathcal{G}$ as many children as there are (non-isomorphic) refinements $\mathcal{G}' = (\mathbb{D}' \times \mathbb{D}', \tau')$ such that $\psi \in \mathsf{obs}(\tau'(x', y'))$ for some point $(x', y') \in \mathbb{D}' \times \mathbb{D}' \setminus \mathbb{D} \times \mathbb{D}$, with $(x, y)\ R\ (x', y')$.

Intuitively, the above refinement strategy guarantees that for every (finite) compass structure $\mathcal{G}$ in the tree-shaped collection $T$ and for every (infinite) compass structure $\mathcal{G}^\star$ over the rationals, if there exists an embedding of $\mathcal{G}$ inside $\mathcal{G}^\star$, then this embedding can be extended to an embedding of a child of $\mathcal{G}$ inside the same structure $\mathcal{G}^\star$. In particular, the only way the procedure can terminate is when no refinement is applicable to the frontier of the tree-shaped collection of compass structures, and hence when $\varphi$ is unsatisfiable over the rationals, as stated by the following proposition.

**Proposition 6.** Let $\varphi$ be an $\mathsf{A\bar{A}B\bar{B}}$ formula. If the procedure of Figure 5 terminates on $\varphi$, then $\varphi$ is unsatisfiable over the class of interval structures with $\mathbb{Q}$ as their temporal domain.

*Proof.* We prove the claim by contraposition, that is, we show that if $\varphi$ has a model over $\mathbb{Q}$, then the procedure does not terminate. Thanks to Proposition 1, we can assume that there exists an infinite compass structure $\mathcal{G}^\star = (\mathbb{Q} \times \mathbb{Q}, \tau^\star)$ such that the formula $\tilde{\varphi} = \langle \mathsf{G} \rangle \varphi$ occurs positively in every atom of it. We proceed as follows. At the beginning of the $i$-th iteration of the while-loop of the procedure, we will make use of $\mathcal{G}^\star$ to identify a suitable leaf $\mathcal{G}_i = (\mathbb{D}_i \times \mathbb{D}_i, \tau_i)$ in the tree-shaped collection $T_i$, which is stored in the variable $T$, that can be embedded inside it. At the same time, by exploiting an inductive argument, we will prove that the content of the queue $Q$ is non-empty and it contains at least one object that can be used to refine $\mathcal{G}_i$, that is, a pair $(z, \mathsf{min})$, with $z = \min(\mathbb{D}_i)$, a pair $(z, \mathsf{max})$, with $z = \max(\mathbb{D}_i)$, a pair $(z_1, z_2)$, with $z_1, z_2$ consecutive elements of $\mathbb{D}$, or a triple $((x,y), R, \psi)$, with $(x,y) \in \mathbb{D}_i \times \mathbb{D}_i$ and $\psi$ being a request of $(x,y)$ along direction $R$ which is not yet fulfilled in $\mathcal{G}_i$. This last invariant enables the inductive argument and, as a by-product, it implies that the procedure never terminates.

As for the inductive base, at the beginning of the first iteration of the while-loop, the frontier of $T$ consists of all and only the singleton compass structures that feature the formula $\tilde{\varphi}$. We arbitrarily choose any such structure whose atom appears also in $\mathcal{G}^\star$ and we denote it by $\mathcal{G}_1 = (\mathbb{D}_1 \times \mathbb{D}_1, \tau_1)$. By construction, $\mathcal{G}_1$ can be embedded inside $\mathcal{G}^\star$. Moreover, it is easy to check that the queue $Q$ contains, among others, the objects $(z, \mathsf{min})$ and $(z, \mathsf{max})$, with $\mathbb{D}_1 = \{z\}$.

As for the inductive step, we assume that, at the beginning of the $i$-th iteration of the while-loop, we have identified a leaf $\mathcal{G}_i$ of $T_i$ that can be embedded inside $\mathcal{G}^\star$, and that the queue $Q$ contains at least one object that, once processed, will induce a proper refinement of $\mathcal{G}_i$. We distinguish some cases depending on the type of object that is removed from the head of $Q$ during such an iteration:

1. If the object that is removed from $Q$ does not affect $\mathcal{G}_i$, that is, if at the end of the $i$-th iteration $\mathcal{G}_i$ is still a leaf of $T_i$, then we simply let $\mathcal{G}_{i+1} = (\mathbb{D}_{i+1} \times \mathbb{D}_{i+1}, \tau_{i+1}) = \mathcal{G}_i$. By the inductive hypothesis, $Q$ still contains an object that can be used to refine $\mathcal{G}_{i+1}$.
2. Otherwise, suppose that the object removed from $Q$ leads, during the execution of the $i$-th iteration, to some refinements of $\mathcal{G}_i = (\mathbb{D}_i \times \mathbb{D}_i, \tau_i)$, say, $\mathcal{G}_{i,1}, \ldots, \mathcal{G}_{i,n}$, with $\mathcal{G}_{i,j} = (\mathbb{D}_{i,j} \times \mathbb{D}_{i,j}, \tau_{i,j})$ for all $1 \leq i \leq n$. First, we observe that the temporal domain $\mathbb{D}_{i,j}$ of each refinement $\mathcal{G}_{i,j}$ is obtained from the temporal domain $\mathbb{D}_i$ of $\mathcal{G}_i$ by inserting a single new element $z'_{i,j}$ with some specific relative order. Moreover, it can be easily seen that the embedding relation of $\mathcal{G}_i$ inside $\mathcal{G}^\star = (\mathbb{Q} \times \mathbb{Q}, \tau^\star)$ can be described by two order-preserving functions $f_i, g_i : \mathbb{D}_i \to \mathbb{Q}$ such that $\tau_i(x,y) = \tau^\star(f_i(x), g_i(y))$ for all $x, y \in \mathbb{D}_i$. Finally, since $\mathbb{Q}$ is a dense order, we can extend both functions $f_i, g_i$ to two new order-preserving functions $f_{i,j}, g_{i,j} : \mathbb{D}_{i,j} \to \mathbb{Q}$. By a close inspection

to the code of Figure 5 (in particular, to the subroutine *AddRefinements*), one can easily check that there must exist an index $1 \leq j \leq n$ for which we have precisely $\tau_{i,j}(x,y) = \tau^{\star}(f_{i,j}(x), g_{i,j}(y))$ for all points $(x,y) \in \mathbb{D}_{i,j} \times \mathbb{D}_{i,j}$ with $x = z'_{i,j}$ or $y = z'_{i,j}$ It immediately follows that the refined compass structure $\mathcal{G}_{i,j}$, hereafter denoted simply by $\mathcal{G}_{i+1}$, can also be embedded inside $\mathcal{G}^{\star}$. To conclude, it suffices to observe that the call to the subroutine *AddRefinements*, with $\mathcal{G} = \mathcal{G}_i$ and $z' = z'_{i,j}$ as arguments, inserts in $Q$ all new objects that can be used to further refine the structure $\mathcal{G}_{i+1}$, which originate from the addition of the element $z'_{i,j}$ to the domain $\mathbb{D}_i$ of $\mathcal{G}_i$.

This allows us to conclude that the proposed semi-decision procedure does not terminate when $\varphi$ is satisfiable over $\mathbb{Q}$. □

A converse result holds as well: if the procedure does not terminate, then the formula $\varphi$ is satisfied by some compass structure $\mathcal{G}$ with a temporal domain isomorphic to $\mathbb{Q}$. The proof stems on two crucial properties: Konig's Lemma, which implies that every finitely-branching tree with infinitely many nodes contains an infinite path, and the existence of a limit construction for an infinite sequence of finer and finer compass structures.

**Proposition 7.** Let $\varphi$ be an $\mathsf{A\bar{A}B\bar{B}}$ formula. If the procedure of Figure 5 does not terminate on $\varphi$, then there exists a globally fulfilling compass structure $\mathcal{G}^{\star} = (\mathbb{Q} \times \mathbb{Q}, \tau)$ whose atoms contain the observable $\langle \mathsf{G} \rangle \varphi$, and hence $\varphi$ is satisfiable over the class of interval structures with $\mathbb{Q}$ as their temporal domain.

*Proof.* Suppose that the procedure does not terminate on $\varphi$ and, for every for $i \geq 1$, let $T_i$ be the tree-shaped collection of compass structures which is generated by the $i$-th iteration of the while-loop. By construction, the tree $T_{i+1}$ is a proper extension of the tree $T_i$, and it is obtained by $T_i$ by appending a finite number of nodes to some of its leaves. Moreover, the limit of the sequence of trees $T_1, T_2, \ldots$ exists and it is a finitely-branching tree $T$ containing infinitely many nodes. By Konig's Lemma, $T$ features at least one infinite path $\rho$. Let $\mathcal{G}_0, \mathcal{G}_1, \ldots$ be the infinite sequence of compass structures appearing along the path $\rho$, with $\mathcal{G}_i = (\mathbb{D}_i \times \mathbb{D}_i, \tau_i)$ for all $i \geq 0$. We have that $\mathcal{G}_1$ consists of a single point, whose atom contains the observable $\tilde{\varphi} = \langle \mathsf{G} \rangle \varphi$, and, for every $i \geq 1$, $\mathcal{G}_{i+1}$ is a proper refinement of $\mathcal{G}_i$.

Now, let $\mathcal{G}$ be the limit of the sequence $\mathcal{G}_1, \mathcal{G}_2, \ldots$, which is defined as $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$, where $\mathbb{D} = \bigcup_{i \geq 1} \mathbb{D}_i$ and, for all $(x,y) \in \mathbb{D} \times \mathbb{D}$, $\tau(x,y) = \tau_i(x,y)$ for sufficiently large indices $i \geq 1$. Below, we show that $\mathcal{G}$ is a consistent and globally fulfilling compass structure over a temporal domain isomorphic to $\mathbb{Q}$.

Consistency of $\mathcal{G}$ follows easily from the observation that all compass structures along $\rho$ are consistent and that consistency is compatible with limit constructions (it is basically a universal property over pairs of points).

To prove that $\mathcal{G}$ is globally fulfilling, let us consider a generic point $(x,y)$ in $\mathbb{D} \times \mathbb{D}$ and a generic request $\psi \in \mathsf{req}_R(\tau(x,y))$ of it along a direction $R \in \{A, \bar{A}, B, \bar{B}\}$. By construction, there exists an index $i$ such that $(x,y)$ is added during the step that refines $\mathcal{G}_i$ into $G_{i+1}$, that is, $(x,y) \in (\mathbb{D}_{i+1} \times \mathbb{D}_{i+1}) \setminus (\mathbb{D}_i \times \mathbb{D}_i)$. During this refinement step (see the last lines of the subroutine *AddRefinements*), the triple $((x,y),R,\psi)$ is inserted in $Q$. The same triple $((x,y),R,\psi)$ will be eventually removed from $Q$ and processed, say, at the execution of the $j$-th iteration of the while-loop, for some $j > i$. It can be easily checked that, during the refinement of $\mathcal{G}_j$ into $\mathcal{G}_{j+1}$, the request $\psi$ will be fulfilled thanks to the insertion of a point $(x',y')$ such that $(x,y)\ R\ (x',y')$ and $\psi \in \mathsf{obs}(\tau_{j+1}(x',y'))$. As $\mathcal{G}_{j+1}$ is embedded inside $\mathcal{G}$, this proves that every request is fulfilled in $\mathcal{G}$.

In a similar way, we can prove that the temporal domain of $\mathcal{G}$ is isomorphic to $\mathbb{Q}$, that is, it is dense and it contains neither a minimum element nor a maximum one. Each element $z$ that, at the $i$-th iteration of the while-loop, becomes the minimal (resp., maximal) element of the domain $\mathbb{D}_i$ forces an object $(z,\mathsf{min})$ (resp., $(z,\mathsf{max})$) to be inserted in $Q$, and thus it will be eventually replaced, in its role of minimal (resp., maximal) element, by another element in a subsequent refinement step. Analogously, every pair of elements $z, z'$ that, at the $i$-th iteration of the while-loop, become consecutive in the domain $\mathbb{D}_i$ is added to $Q$ and eventually processed and separated by the insertion of a third element between them in a subsequent refinement step. As a consequence, the domain $\mathbb{D}$ of the limit structure $\mathcal{G}$ neither features a minimal (resp., maximal) element nor a pair of consecutive elements.

This allows us to conclude that the $\mathcal{G}$ is a consistent and globally fulfilling compass structure witnessing the satisfiability of $\varphi$ over a temporal domain isomorphic to $\mathbb{Q}$. $\square$

### A.3 Enumerating satisfiable formulas over $\mathbb{Q}$

Here we give the proofs of the results concerning the enumeration of $\mathsf{A\bar{A}B\bar{B}}$ formulas satisfiable over $\mathbb{Q}$.

We begin by describing how we can replace, without loss of generality, the temporal domain $\mathbb{Q}$ with a dense order with endpoints, namely, a linear ordering $\mathbb{D}$ that has minimal and maximal elements. Suppose that a formula $\varphi$ is given that is interpreted over $\mathbb{Q}$. Then this formula can be rewritten into an equi-satisfiable formula $\varphi_{][}$ that is interpreted over a dense order with endpoints $\mathbb{D}$, by simply restricting all quantifications of $\varphi$ to range over intervals satisfying $\langle\bar{\mathsf{A}}\rangle\neg\pi \wedge \langle\mathsf{A}\rangle\neg\pi$, that is, intervals that neither start nor end at the endpoints of $\mathbb{D}$. Symmetrically, every formula $\varphi$ interpreted over a dense order with endpoints $\mathbb{D}$ can be turned into an equi-satisfiable formula $\varphi_{[\,]}$ interpreted over $\mathbb{Q}$, by relativising all quantifications to intervals satisfying $\langle\bar{\mathsf{A}}\rangle\mathsf{closed} \wedge \langle\mathsf{A}\rangle\mathsf{closed}$, where $\mathsf{closed}$ is a fresh propositional letter satisfying $[\mathsf{G}](\mathsf{closed} \rightarrow \pi) \wedge [\mathsf{G}]((\langle\bar{\mathsf{A}}\rangle\mathsf{closed} \wedge \langle\mathsf{A}\rangle\mathsf{closed}) \rightarrow ([\mathsf{B}]\langle\bar{\mathsf{A}}\rangle\mathsf{closed})) \wedge \langle\mathsf{G}\rangle(\mathsf{closed} \wedge [\bar{\mathsf{A}}](\pi \vee [\bar{\mathsf{A}}]\neg\mathsf{closed})) \wedge \langle\mathsf{G}\rangle(\mathsf{closed} \wedge [\mathsf{A}](\pi \vee [\mathsf{A}]\neg\mathsf{closed}))$ – technically speaking, the set of $\mathsf{closed}$-labelled time points defined by the latter formula is a closed convex subset of $\mathbb{Q}$. The above correspondence allows us to reduce satisfiability of an $\mathsf{A\bar{A}B\bar{B}}$ formula $\varphi$ over $\mathbb{Q}$ to the existence of a compass structure $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$, where $\mathbb{D}$ is a dense order with endpoints, that contains the observable $\langle\mathsf{G}\rangle\varphi_{][}$ in all its atoms.

We now turn to proving that full profile trees are correct abstractions of globally fulfilling compass structures. We give separate proofs for the two statements of Proposition 2.

**Proposition 2 (first statement).** *For every globally fulfilling compass structure $\mathcal{G} = (\mathbb{D} \times \mathbb{D}, \tau)$ over a dense order with endpoints $\mathbb{D}$, there is a full profile tree $\mathcal{T} = (T, N, E)$ such that $T$ is a decomposition of $\mathbb{D}$ and, for all nodes $s \in T$, $N(s) = \mathsf{profile}(\mathcal{G}_s)$.*

*Proof.* The proof of this first statement is quite standard, as it amounts at defining an appropriate decomposition $T$ of the domain of the compass structure $\mathcal{G}$. This is done by induction by considering a pair $s = (y_1, y_2)$ that is already defined to be a node of $T$ and by introducing sufficiently many children of $s$ with appropriate coordinates, so as to satisfy all the conditions of Definition 2. We give the full details below.

Let $\mathcal{G} = (\mathbb{D}\times\mathbb{D}, \tau)$ be a globally fulfilling compass structure over a dense order with endpoints $\mathbb{D}$. Below we define a suitable decomposition $T$ of the domain $\mathbb{D}$ and the corresponding labeling for the profile tree $\mathcal{T} = (T, N, E)$ by exploiting an induction.

In the base case, we simply let the root of $T$ be the pair $s_0 = (\min(\mathbb{D}), \max(\mathbb{D}))$ and let $N(s_0)$ be the profile of the outermost slice $\mathcal{G}_{s_0}$ of $\mathcal{G}$. We then observe that $N(s_0)$ contains only one occurrence of a pair $(F, G)$, with $F = \tau(\min(\mathbb{D}), \min(\mathbb{D}))$ and $G = \tau(\min(\mathbb{D}), \max(\mathbb{D}))$, plus some occurrences of pairs of the form $(\varnothing, G)$, with $G = \tau(x, \max(\mathbb{D}))$ and $x > \min(\mathbb{D})$. Moreover, for all pairs $(F, G) \in N(s_0)$, we have that $\mathsf{req}_{\bar{B}}(G) = \varnothing$ since $\mathcal{G}$ is a globally fulfilling compass structure. It follows that, if the tree $\mathcal{T} = (T, N, E)$ that will result in the limit construction is a profile tree, then it is also a *full* profile tree.

For the induction step, let $s = (y_1, y_2)$ be an already-defined node of $T$ (a leaf of the current partial tree) and $N(s) = \mathsf{profile}(\mathcal{G}_s)$ be the associated profile. We define the children $s_1, \ldots, s_n$ of $s$ in $T$ and the multisets $E(s), N(s_1), \ldots, N(s_n)$ as follows. The number $n$ of children of $s$, as well as their coordinates, will depend on the arrangement of the points in the slice $\mathcal{G}_s$ that witness the pairs of the profile $N(s)$. In particular, to define these children we need to distinguish in $N(s)$ the pairs that occur with finite multiplicity from those that occur with infinite multiplicity.

Since $N(s) = \mathsf{profile}(\mathcal{G}_s)$, for each pair $(F, G)$ with $N(s)(F, G) < \infty$, there exist exactly $N(s)(F, G)$ pairwise distinct coordinates $x \in \mathbb{D}$ such that $\tau(x, y_1) = F$ and $\tau(x, y_2) = G$. We collect all these coordinates into a set $X_{F,G}$ of finite cardinality $N(s)(F, G)$, for each pair $(F, G)$ with $N(s)(F, G) < \infty$. Moreover, for each pair $(F, G)$ with $N(s)(F, G) = \infty$, we choose a coordinate $x^{\infty}_{F,G}$ from the set $\mathbb{D} \setminus \bigcup_{N(s)(F,G)<\infty} X_{F,G}$ in such a way that $\tau(x^{\infty}_{F,G}, y_1) = F$ and $\tau(x^{\infty}_{F,G}, y_2) = G$. We then define

$$X \;=\; \bigcup_{N(s)(F,G)<\infty} X_{F,G} \quad\cup \bigcup_{N(s)(F,G)=\infty} \{x^{\infty}_{F,G}\}$$

Next, we consider the requests that originate from the points of the slice $\mathcal{G}_s$ that intercept the coordinates of $X$. More precisely, for each $x \in X$ and each $\psi \in \mathsf{req}_{\bar{B}}(\tau(x, y_1)) \setminus \mathsf{req}_{\bar{B}}(\tau(x, y_2))$, we choose a point $(x, y_{x,\psi})$, with $y_1 < y_{x,\psi} < y_2$, such that $\psi \in \mathsf{obs}(\tau(x, y_{x,\psi}))$ (recall that the compass structure $\mathcal{G}$ is globally fulfilling). Similarly, for each $x \in X$ and each $\psi' \in \mathsf{req}_B(\tau(x, y_2)) \setminus \mathsf{req}_B(\tau(x, y_1))$, we choose a point $(x, y^{x,\psi'})$, with $y_1 < y^{x,\psi'} < y_2$, such that $\psi' \in \mathsf{obs}(\tau(x, y^{x,\psi'}))$.

We collect all coordinates $y_{x,\psi}$ and $y^{x,\psi'}$ into a finite set $Y$. Finally, we define the set

$$Z \ = \ \{y_1, y_2\} \ \cup \ Y \ \cup \ \{x \in X \ : \ y_1 < x < y_2\}$$

and we order its elements as follows: $y_1 = z_1 < z_2 < \ldots < z_n < z_{n+1} = y_2$. Without loss of generality, we can assume that $n \geq 2$: if this was not the case, then we could have chosen an arbitrary pair $(\varnothing, G)$ with infinite multiplicity in $N(s)$ (there exists at least one such pair) and, instead of a single coordinate $x^{\infty}_{F,G}$, we could have taken two such coordinates.

The children of the node $s$ in $T$ are precisely defined as the $n$ pairs $s_1 = (z_1, z_2)$, …, $s_n = (z_n, z_{n+1})$. We observe that $|X| \leq \sum_{N(s)(F,G)<\infty} N(s)(F,G) + \sum_{N(s)(F,G)=\infty} 1$ and hence $|Z| \leq 2 + 2|X| \cdot |\mathsf{closure}(\varphi)| + |X|$. From this, by considering $\varphi$ as a fixed parameter, we derive that the number $n$ of children of $s$ is $\mathcal{O}\big( \sum_{N(s)(F,G)<\infty} N(s)(F,G) \ + \ \sum_{N(s)(F,G)=\infty} 1 \big)$.

The labelling on node $s$ is defined as follows. We let $E(s)$ be the multiset that contains any $(n+1)$-tuple of atoms $\bar{F} = (F_1, \ldots, F_{n+1}) \in (\mathsf{atoms}(\varphi) \uplus \{\varnothing\})^{n+1}$ with the following multiplicity:

$$E(s)(\bar{F}) \ = \ \big|\{x \in \mathbb{D} \ : \ \tau(x, z_1) = F_1, \ \ \ldots, \ \tau(x, z_{n+1}) = F_{n+1}\}\big|.$$

In addition, we define $N(s_i) = E(s)|_{i,i+1}$ for all $1 \leq i \leq n$.

It remains to verify that all the conditions of Definition 2 are satisfied for a generic node $s \in T$. The first condition **(profile-match)** is clearly satisfied, since $s$ has $n \geq 2$ children $s_1, \ldots, s_n$, $E(s)|_{1,n+1} = N(s)$, and $E(s)|_{i,i+1} = N(s_i)$ for all $1 \leq i \leq n$. To verify the second condition **(profile-finite-req)**, we consider a tuple $(F_1, \ldots, F_{n+1})$ in $E(s)$ such that $F_1 \neq \varnothing$ and $N(s)(F_1, F_{n+1}) < \infty$. We first observe that the inclusions $\mathsf{req}_{\bar{B}}(F_1) \supseteq \bigcup_{2\leq i\leq n+1} \mathsf{obs}(F_i) \ \cup \ \mathsf{req}_{\bar{B}}(F_{n+1})$ and $\mathsf{req}_{B}(F_{n+1}) \supseteq \bigcup_{1\leq i\leq n} \mathsf{obs}(F_i) \ \cup \ \mathsf{req}_{B}(F_1)$ follow trivially from the fact that $N(s) = \mathsf{profile}(\mathcal{G}_s)$ and from the consistency conditions satisfied by the atoms in compass structure $\mathcal{G}$. As for the converse inclusions, let $\psi$ be a request in $\mathsf{req}_{\bar{B}}(F_1) \setminus \mathsf{req}_{\bar{B}}(F_{n+1})$. We know from the constructions above that the finite set $X$ contains all the (finitely many) coordinates $x \in \mathbb{D}$ such that $(x, z_i)$ is an $F_i$-labelled point for all $1 \leq i \leq n+1$. In particular, for all such coordinates $x \in X$, there exists $y_{x,\psi} \in Y$ such that $\psi \in \mathsf{obs}(\tau(x, y_{x,\psi}))$. Again by construction, we know that for some $2 \leq i \leq n+1$, $y_{x,\psi} = z_i$ and hence $\psi \in \mathsf{obs}(F_i)$. This shows that $\mathsf{req}_{\bar{B}}(F_1) \subseteq \bigcup_{2\leq i\leq n+1} \mathsf{obs}(F_i) \ \cup \ \mathsf{req}_{\bar{B}}(F_{n+1})$. Symmetric arguments can be used to prove that $\mathsf{req}_{B}(F_{n+1}) \subseteq \bigcup_{1\leq i\leq n} \mathsf{obs}(F_i) \ \cup \ \mathsf{req}_{B}(F_1)$. The third condition **(profile-infinite-req)** is proved in a similar way. We now verify the last condition **(profile-dummy)**. Let $(\varnothing, G)$ be a pair that occurs in $N(s)$. We know from the previous constructions that the set $X$ contains at least one coordinate $x$ for which $\tau(x, y_1) = \varnothing$ and $\tau(x, y_2) = G$. This coordinate $x$ coincides with one of the coordinates $z_i$ that we collected in $Z$. It follows that the multiset $E(s)$ contains an occurrence of a tuple $(F_1, \ldots, F_{n+1})$, where $F_1 = \tau(x, z_1)$, …, $F_{n+1} = \tau(x, z_{n+1})$. In particular, we have that $F_{n+1} = \tau(x, z_{n+1}) = \tau(x, y_2) = G$ and that $F_i = \tau(x, z_i) = \tau(x, x)$ is a $\pi$-atom. This proves that $\mathcal{T} = (T, N, E)$ is a profile tree. Previous arguments imply that it is also a *full* profile tree. Finally, it follows by construction that $N(s) = \mathsf{profile}(\mathcal{G}_s)$ for all nodes $s \in T$. ☐

**Proposition 2 (second statement).** *For every full profile tree $\mathcal{T} = (T, N, E)$, with $T$ decomposition of some dense order with endpoints $\mathbb{D}$, there is a globally fulfilling compass structure $\mathcal{G} = (\mathbb{D}' \times \mathbb{D}', \tau)$, with $\mathbb{D}' \subseteq \mathbb{D}$ dense order with endpoints, such that* $\mathsf{set}(\mathsf{profile}(\mathcal{G}_s)) = \mathsf{set}(N(s))$ *for all* $s \in T$.

The proof of this second statement is more technical. The general idea is to construct the domain $\mathbb{D}'$ that underlies the compass structure generated from $\mathcal{T} = (T, N, E)$ as the union of the sets $\{y_1, y_2\}$ over all nodes $s = (y_1, y_2) \in T$ (note that it may happen that $\mathbb{D}' \subsetneq \mathbb{D}$). The definition of the labelling $\tau : \mathbb{D}' \times \mathbb{D}' \to \mathsf{atoms}(\varphi) \uplus \{\varnothing\}$ exploits several conditions from Definition 1 and Definition 2, plus some counting arguments on multisets. A crucial step consists of extending in an appropriate way the matching conditions of Definition 2 across several nodes that lie at the same level of the tree $\mathcal{T}$. Roughly speaking, this step can be though of as a "composition" of the multisets $E(s_1)$, ..., $E(s_{n_\ell})$ that are associated with the nodes $s_1, \ldots, s_{n_\ell}$ at level $\ell$. However, the composition operation is non-deterministic and committing to certain choices can sometimes result in a compass structure that is not globally fulfilling. The correct composition operation is captured by Lemma 5 below.

Before proving the lemma, however, we need to introduce other preliminary definitions and results, which will be used later to iteratively refine multisets of tuples. We begin by introducing a strengthening of the containment relation on multisets, denoted by $\sqsubseteq$ and defined as follows:

$$M \sqsubseteq N \qquad \text{iff} \qquad \begin{cases} \mathsf{set}(M) = \mathsf{set}(N) \\ M(\bar{F}) \leq N(\bar{F}) & \text{for all tuples } \bar{F} \text{ of atoms} \\ M(\bar{F}) = N(\bar{F}) & \text{for all } \pi\text{-tuples } \bar{F}. \end{cases}$$

It is worth pointing out that, unlike the containment order, $\sqsubseteq$ is not a *well* partial order, namely, it may admit infinite anti-chains. However, $\sqsubseteq$ becomes a well partial order when restricted to multisets containing at most one occurrence of each $\pi$-tuple (examples of such multisets are those associated with the nodes in a profile tree).

We then introduce an operation of "insertion" for multisets:

**Definition 5.** Given two multisets $L$ and $E$, consisting of occurrences of $(m+1)$-tuples and $(n+1)$-tuples, respectively, and given a position $1 \leq j \leq m$, we define an *insertion of $E$ into $L$ at position $j$* as a function $f : L \to E$ that satisfies the following conditions:

- $f$ maps any occurrence of a tuple $(G_1, \ldots, G_{m+1})$ in $L$ to an occurrence of a tuple $(F_1, \ldots, F_{n+1})$ in $E$ such that $F_1 = G_j$ and $F_{n+1} = G_{j+1}$;
- $f$ is surjective, namely, every occurrence of a tuple in $E$ is the image via $f$ of some occurrence of a tuple in $L$;
- the restriction of $f$ that ranges over the occurrences of the $\pi$-tuples of $E$ is injective, namely, there exist no pairs of distinct occurrences of tuples in $L$ that are mapped via $f$ to the same occurrence of a $\pi$-tuple of $E$.

It is easy to see that the existence of an insertion of $E$ into $L$ at position $j$ is equivalent to the relationship

$$E|_{1,n+1} \sqsubseteq L|_{j,j+1}.$$

Next, we establish the following rather simple lemma.

**Lemma 4.** Let $m, n > 0$, $1 \leq j \leq m$, let $L$ and $E$ be multisets of $(m+1)$-tuples and $(n+1)$-tuples, respectively, and let $f$ be an insertion of $E$ into $L$ at position $j$. There is a multiset $L'$ of $(m+n)$-tuples such that

$$\begin{aligned} L &= L'|_{\{1,\dots,j\}\cup\{j+n,\dots,m+n\}} \\ E &\sqsubseteq f(L) = L'|_{\{j,\dots,j+n\}}. \end{aligned}$$

*Proof.* In this proof it is convenient to think of the insertion $f$ as a multi-relation, namely, as a multiset $R$ that maps every pair $(\bar{G}, \bar{F})$ consisting of an $(m+1)$-tuple $\bar{G} \in \mathsf{set}(L)$ and an $(n+1)$-tuple $\bar{F} \in \mathsf{set}(E)$ to the number of occurrences of $\bar{G}$ in $L$ that are mapped via $f$ to some occurrences of $\bar{F}$ in $E$. In particular, we have $\sum_{\bar{F}} R(\bar{G}, \bar{F}) = L(\bar{G})$ and $\sum_{\bar{G}} R(\bar{G}, \bar{F}) = f(L)(\bar{F})$.

We begin by defining $L'$ as a multiset of $(m+n)$-tuples of the form $(G_1, \dots, G_j, F_2, \dots, F_n, G_{j+1}, \dots, G_{m+1})$. More precisely, each tuple $(G_1, \dots, G_j, F_2, \dots, F_n, G_{j+1}, \dots, G_{m+1})$ occurs in $L'$ with multiplicity exactly equal to $R(\bar{G}, \bar{F})$, where $\bar{G} = (G_1, \dots, G_j, G_{j+1}, \dots, G_{m+1})$ and $\bar{F} = (G_j, F_2, \dots, F_n, G_{j+1})$.

By construction, every tuple in $L$ can be obtained from a tuple in $L'$ by removing the intermediate components $F_2, \dots, F_n$ and, vice versa, every tuple in $L'$ can be obtained from a tuple in $L$ by inserting appropriate components, determined by $G_1, \dots, G_{m+1}$ using $f$. This amounts to say that there is a bijection $g$ from $L$ to $L'$ that maps any occurrence of a tuple $(G_1, \dots, G_{m+1})$ to a distinguished occurrence of a tuple $(G_1, \dots, G_j, F_2, \dots, F_m, G_{j+1}, \dots, G_{n+1})$; the inverse of $g$ is nothing but the projection that hides the intermediate components $F_2, \dots, F_m$:

$$L = L'|_{\{1,\dots,j\}\cup\{j+n,\dots,m+n\}}.$$

To prove the second property, we recall that the function $f$ is surjective, and, furthermore, it is injective when restricted to range over occurrences of $\pi$-tuples of $E$. This immediately implies $E \sqsubseteq f(L)$. Finally, the fact that $f(L) = L'|_{j,\dots,j+n}$ follows easily from the definition of $R$ and $L'$, namely, for all tuples $\bar{F} = (G_j, F_2, \dots, F_n, G_{j+1})$, we have:

$$\begin{aligned} L'|_{\{j,\dots,j+n\}}(\bar{F}) &= \sum_{\substack{G_1,\dots,G_{j-1} \\ G_{j+2},\dots,G_{m+1}}} L'(G_1, \dots, G_j,\, F_2, \dots, F_n,\, G_{j+1}, \dots, G_{m+1}) \\ &= \sum_{\bar{G}=(G_1,\dots,G_{m+1})} R(\bar{G}, \bar{F}) \\ &= f(L)(\bar{F}). \end{aligned}$$

□

Below, we prove the crucial lemma underlying the translation from full profile trees to globally fulfilling compass structures. According this lemma, for every

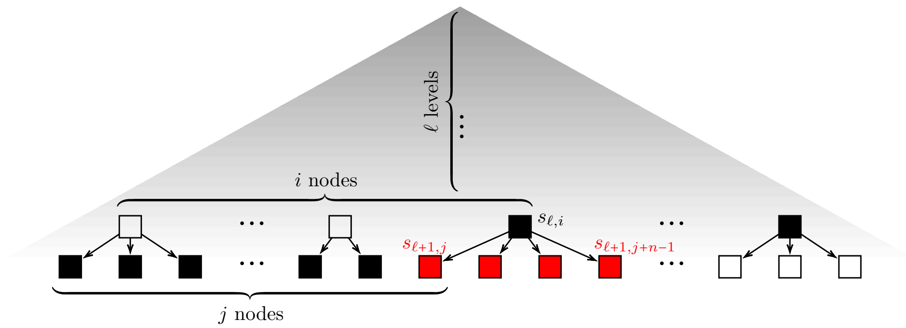


**Fig. 6.** Graphical representation of the refinement step from $L(\ell, i-1)$ to $L(\ell, i)$.

$\ell \in \mathbb{N}$ and $1 \le j \le n_\ell$, the multiset $L(\ell)$ contains exactly one occurrence of a tuple of the form $(G_1, \dots, G_{n_\ell})$, where $G_j$ is a $\pi$-atom (having more than one occurrence would violate the fourth condition of Definition 1 for the profile $N(s_{\ell,j})$). The intuition is that the $\pi$-atom $G_j$ in the above tuple represents the type of the singleton interval $[y_{\ell,j}, y_{\ell,j}]$. Accordingly, the labelling $\tau$ of the compass structure that we construct from $\mathcal{T}$ associates with each point $p = (x, y)$, where $x = y_{\ell,j}$ and $y = y_{\ell,i}$, the $i$-th atom $G_i$ in the above tuple. Once the compass structure $\mathcal{G} = (\mathbb{D}' \times \mathbb{D}', \tau)$ is defined, it remains to check that it is indeed consistent and globally fulfilling.

**Lemma 5.** Let $\mathcal{T} = (T, N, E)$ be a full profile tree and, for each $\ell \in \mathbb{N}$, let $s_{\ell,1} = (y_{\ell,1}, y_{\ell,2})$, ..., $s_{\ell,n_\ell} = (y_{\ell,n_\ell}, y_{n_\ell+1})$ be the nodes, listed from left to right, that lie at level $\ell$ in the decomposition tree $T$. One can construct a function $L$ that maps any number $\ell \in \mathbb{N}$ to a multiset of $(n_\ell + 1)$-tuples of atoms such that:

– $N(s_{0,1}) = L(0)$ and $N(s_{\ell,i}) \sqsubseteq L(\ell)|_{i,i+1}$ for all $1 \le i \le n_\ell$;
– $E(s_{\ell,i}) \sqsubseteq L(\ell+1)|_{\{j,\dots,j+n\}}$, where $s_{\ell+1,j}$, ..., $s_{\ell+1,j+n-1}$ are the children of $s_{\ell,i}$;
– $L(\ell)|_{\{i_1,\dots,i_h\}} = L(\ell')|_{\{i'_1,\dots,i'_h\}}$ for all $\ell' \ge \ell$, $1 \le i_1 < \dots < i_h \le n_\ell + 1$, and $1 \le i'_1 < \dots < i'_h \le n_{\ell'} + 1$, with $y_{\ell,i_1} = y_{\ell',i'_1}$, ..., $y_{\ell,i_h} = y_{\ell',i'_h}$;
– $\mathsf{req}_{\bar{B}}(G_k) = \bigcup_{k<h\le n_\ell+1} \mathsf{obs}(G_h)$ for all $\pi$-tuples $(G_1, \dots, G_{n_\ell+1}) \in L(\ell)$ and all $1 \le k \le n_\ell + 1$, with $G_k \ne \emptyset$;
– $\mathsf{req}_{B}(G_k) = \bigcup_{1\le h<k} \mathsf{obs}(G_h)$ for all $\pi$-tuples $(G_1, \dots, G_{n_\ell+1}) \in L(\ell)$ and all $1 \le k \le n_\ell + 1$, with $G_k \ne \emptyset$.

*Proof.* To start with, we define $L(0)$ as the profile $N(s_{0,1})$ associated with the root $s_{0,1}$ of the profile tree $\mathcal{T} = (T, N, E)$. To define $L(\ell+1)$, for every $\ell \ge 0$, we construct some multisets $L(\ell, i)$ parametrised by two indices $\ell$ and $i$, with $\ell \ge 0$ and $0 \le i \le n_\ell$. Intuitively, each multiset $L(\ell, i)$ represents a refinement of the multiset $L(\ell)$ with the successors of the first $i$ nodes at level $\ell$ in the profile tree $\mathcal{T}$ (see Figure 6). In particular, we have $L(\ell, n_\ell) = L(\ell+1)$, for all $\ell \in \mathbb{N}$.

The definition of the multisets $L(\ell, i)$ will exploit a double induction on the parameters $\ell$ and $i$, as briefly outlined in the following. We let by convention $L(\ell+1, 0) = L(\ell, n_\ell)$, for all $\ell \in \mathbb{N}$. Then, given $\ell \in \mathbb{N}$ and $1 \le i \le n_\ell$ (resp., $\ell > 0$

and $i = 0$), we construct the multiset $L(\ell, i)$ via an application of Lemma 4 that takes into account the previously defined multiset $L(\ell, i-1)$ (resp., $L(\ell-1, n_{\ell-1})$) and the multiset $E(s_{\ell,i})$ associated with the $i$-th node of $\mathcal{T}$ at level $\ell$. To let the induction go through, we will also enforce the following invariants for all indices $\ell \in \mathbb{N}$ and $0 \leq i, i' \leq n_\ell$:

1. if $i' \leq i$ and $s_{\ell+1,k}$ is a child of $s_{\ell,i'}$, then $N(s_{\ell+1,k}) \sqsubseteq L(\ell, i)|_{k,k+1}$;
2. if $i' > i$ and $s_{\ell+1,k}$ is the last child of $s_{\ell,i}$, then $N(s_{\ell,i'}) \sqsubseteq L(\ell, i)|_{k+(i'-i),k+(i'-i)+1}$;
3. if $(G_1, \ldots, G_{m+1})$ is a $\pi$-tuple in $L(\ell, i)$, then for all $1 \leq k \leq m+1$ either $G_k \neq \varnothing$ or $\mathsf{req}_{\bar{B}}(G_k) = \bigcup_{k<h\leq m+1} \mathsf{obs}(G_h)$;
4. if $(G_1, \ldots, G_{m+1})$ is a $\pi$-tuple in $L(\ell, i)$, then for all $1 \leq k \leq m+1$, either $G_k = \varnothing$ or $\mathsf{req}_B(G_k) = \bigcup_{1\leq h<k} \mathsf{obs}(G_h)$.

Once all $L(\ell, i)$ are defined, we will verify that the multisets $L(\ell)$ $(= L(\ell, 0))$ satisfy the conditions of the lemma.

The base case of the inductive construction holds when $\ell = 0$ and $i = 0$, and it consists of defining $L(0,0)$ as $L(0)$. This definition clearly satisfies the invariants I1–I4.

In the inductive step, we fix $\ell \in \mathbb{N}$ and $1 \leq i \leq n_\ell$, we assume $L(\ell, i-1)$ to be defined as a multiset of $(m+1)$-tuples (recall that $L(\ell, 0) = L(\ell-1, n_{\ell-1})$), and we construct the next multiset $L(\ell, i)$.

To this end, we consider the node $s_{\ell,i}$ together with its children (ordered from left to right), say, $s_{\ell+1,j}$, $\ldots$, $s_{\ell+1,j+n-1}$, for some $j \geq 1$ and some $n \geq 2$. By definition, $E(s_{\ell,i})$ consists of $(n+1)$-tuples of atoms. The definition of profile tree and invariant I2 imply that

$$E(s_{\ell,i})|_{1,n+1} \;=\; N(s_{\ell,i}) \;\sqsubseteq\; L(\ell, i-1)|_{j,j+1}.$$

This means that there is an *insertion* $f$ of $E(s_{\ell,i})$ into $L(\ell, i-1)$ at position $j$.

We now state an important property of the function $f$. Consider a generic tuple $(F_1, \ldots, F_{n+1}) \in E(s_{\ell,i})$, with $F_1 \neq \varnothing$, and suppose that the pair $(F_1, F_{n+1})$ has *finite* multiplicity in $N(s_{\ell,i})$. By condition **(profile-finite-req)** of Definition 2, it immediately follows that $\mathsf{req}_{\bar{B}}(F_1) = \bigcup_{1<h\leq n+1} \mathsf{obs}(F_h) \cup \mathsf{req}_{\bar{B}}(F_{n+1})$ and $\mathsf{req}_B(F_{n+1}) = \bigcup_{1\leq h<n+1} \mathsf{obs}(F_h) \cup \mathsf{req}_B(F_1)$. This property will be used later to prove that the resulting multiset $L(\ell, i)$ satisfies the invariants I3 and I4. As a matter of fact, to prove these invariants we need to enforce an additional constraint on $f$, which concerns those pairs $(F_1, F_{n+1})$, with $F_1 \neq \varnothing$, that have *infinite* multiplicity in $N(s_{\ell,i})$. Let $(F_1, F_{n+1})$ be any such pair (if there is none, then the assumption that follows is simply vacuous). Condition **(profile-infinite-req)** of Definition 2 requires the existence of a tuple $\bar{F} = (F_1, \ldots, F_{n+1})$ in $E(s_{\ell,i})$ such that $\mathsf{req}_{\bar{B}}(F_1) = \bigcup_{1<h\leq n+1} \mathsf{obs}(F_h) \cup \mathsf{req}_{\bar{B}}(F_{n+1})$ and $\mathsf{req}_B(F_{n+1}) = \bigcup_{1\leq h<n+1} \mathsf{obs}(F_h) \cup \mathsf{req}_B(F_1)$. We fix any such tuple $\bar{F}$, that is, we commit to a specific choice for the atoms $F_2, \ldots, F_n$, and we consider the occurrences of tuples in $L(\ell, i-1)$ that are mapped by $f$ to occurrences of $\pi$-tuples of $E(s_{\ell,i})$, namely, we let $E_\pi(s_{\ell,i})$ be the restriction of the multiset $E(s_{\ell,i})$ to its $\pi$-tuples and we define $L_\pi(\ell, i-1) = f^{-1}(E_\pi(s_{\ell,i}))$ (clearly, $L_\pi(\ell, i-1)$ is contained, as a multiset, in $L(\ell, i-1)$). Since $f$ is an insertion, its restriction to $L_\pi(\ell, i-1)$ is injective. Moreover, by the definitions of profile and profile tree (specifically, by the fourth item of Definition 1 and the first item of Definition 2), all multiplicities

of $E_\pi(s_{\ell,i})$ must be 1. By the Pigeonhole Principle, it follows that all multiplicities of $L_\pi(\ell,i-1)$ are finite. Since the sum of the multiplicities of the tuples $(G_1,\ldots,G_{m+1})$ in $L(\ell,i-1)$ such that $G_j = F_1$ and $G_{j+1} = F_{n+1}$ is infinite (recall that, by assumption, $N(s_{\ell,i})(F_1,F_{n+1}) = \infty$ and $N(s_{\ell,i}) \sqsubseteq L(\ell,i-1)|_{j,j+1}$), we can enforce $f$ to map all occurrences of tuples of $L_\pi(\ell,i-1)$ of the form $(G_1,\ldots,G_{m+1})$, with $G_j = F_1$ and $G_{j+1} = F_{n+1}$, to the *same occurrence* of the tuple $\bar{F}$, and this can be done without violating the property of being an insertion.

We are now ready to define the multiset $L(\ell,i)$ recursively from $L(\ell,i-1)$, $E(s_{\ell,i})$, and $f$, by exploiting Lemma 4, in such a way that

$$\begin{aligned} L(\ell,i-1) &= L(\ell,i)|_{\{1,\ldots,j\}\cup\{j+n,\ldots,m+n\}} \\ E(s_{\ell,i}) &\sqsubseteq f(L(\ell,i-1)) = L(\ell,i)|_{\{j,\ldots,j+n\}} \end{aligned}$$

(recall that $j$ is the index that identifies the first child $s_{\ell+1,j}$ of $s_{\ell,i}$ at level $\ell+1$).

Let us verify that the defined multiset $L(\ell,i)$ satisfies all the invariants I1–I4. The first invariant holds for all $i' < i$ thanks to the inductive hypothesis. Moreover, it also holds for $i' = i$ and all $k = j+h$, with $1 \le h \le n$, thanks to the fact that $N(s_{\ell+1,k}) = E(s_{\ell,i})|_{h,h+1} \sqsubseteq L(\ell,i)|_{\{k,k+1\}}$. Similarly, the second invariant holds for all $i' > i$ thanks to the inductive hypothesis and the fact that $L(\ell,i-1)|_{\{j+1,\ldots,m+1\}} = L(\ell,i)|_{\{j+n,\ldots,m+n\}}$. As for the invariant I3, we consider a $\pi$-tuple $(G_1,\ldots,G_{m+n})$ in $L(\ell,i)$ and a non-dummy atom $G_k$ in it, for some $1 \le k \le m+n$. We distinguish two cases, depending on which among the two sub-sequences $(G_1,\ldots,G_j,G_{j+n},\ldots,G_{m+n})$ and $(G_{j+1},\ldots,G_{j+n-1})$ is a $\pi$-tuple. If the former sub-sequence is a $\pi$-tuple, then it also occurs in $L(\ell,i-1)$ and hence the invariant $\mathsf{req}_{\bar{B}}(G_k) = \bigcup_{k<h\le m+n}\mathsf{obs}(G_h)$ follows easily from the inductive hypothesis. Otherwise, we know that $(G_j,\ldots,G_{j+n})$ is also a $\pi$-tuple that occurs in $E(s_{\ell,i})$ ($\sqsubseteq f(L(\ell,i-1))$). Moreover, thanks to the properties satisfied by the insertion $f$, we derive that $\mathsf{req}_{\bar{B}}(G_k) = \bigcup_{k<h\le j+n}\mathsf{obs}(G_h) \cup \mathsf{req}_{\bar{B}}(G_{j+n})$. Using the fact that $\mathsf{req}_{\bar{B}}(G_{j+n}) = \bigcup_{j+n<h\le m+n}\mathsf{obs}(G_h)$, we conclude that $\mathsf{req}_{\bar{B}}(G_k) = \bigcup_{k<h\le m+n}\mathsf{obs}(G_h)$. Similar arguments can be used to prove that the invariant I4 holds for any $\pi$-tuple $(G_1,\ldots,G_{m+n})$ in $L(\ell,i)$ and any atom $G_k$ in it.

Now that we have defined the multisets $L(\ell,i)$, we let $L(\ell) = L(\ell,0)$ for all $\ell \in \mathbb{N}$. To complete the proof, we must show that the multisets $L(\ell)$ satisfy the properties stated in the lemma. First of all, we observe that the last two properties follow easily from the invariants I3 and I4 when we let $i = 0$. Moreover, if $s_{\ell+1,j},\ldots,s_{\ell+1,j+n-1}$ are the children of the $i$-th node $s_{\ell,i}$ at level $\ell$ in $\mathcal{T}$, then the second property follows easily by construction:

$$E(s_{\ell,i}) \sqsubseteq L(\ell,i)|_{\{j,\ldots,j+n\}} = L(\ell,n_\ell)|_{\{j,\ldots,j+n\}} = L(\ell+1)|_{\{j,\ldots,j+n\}}.$$

Similarly, if $s_{\ell+1,k}$ is the $h$-th children of $s_{\ell,i}$, then $k = j+h-1$, and hence the first property also holds:

$$\begin{aligned} &N(s_{0,1}) = L(0) \\ &N(s_{\ell+1,k}) = E(s_{\ell,i})|_{h,h+1} \sqsubseteq \big(L(\ell+1)|_{\{j,\ldots,j+n\}}\big)\big|_{h,h+1} = L(\ell+1)|_{k,k+1}. \end{aligned}$$

It remains to verify the third property. For this, we consider the coordinates $(y_{\ell,i}, y_{\ell,i+1})$ associated with each node $s_{\ell,i}$ in $\mathcal{T}$ and we let $I_\ell = \{i : 1 \le i \le n_\ell + 1\}$ and $Y_\ell = \{y_{\ell,i} : i \in I_\ell\}$, for all $\ell \in \mathbb{N}$. Moreover, we denote tuples of indices from $I_\ell$ by $\bar{i} = (i_1, \ldots, i_h)$ and, by a slight abuse of notation, we denote the corresponding tuples of coordinates by $y(\ell, \bar{i}) = (y_{\ell,i_1}, \ldots, y_{\ell,i_h})$. The property to be proved can be rewritten as follows:

$$\begin{aligned} &\forall \ell \le \ell' \quad \forall \bar{i} \in I_\ell \times \ldots \times I_\ell \quad \forall \bar{i}' \in I_{\ell'} \times \ldots \times I_{\ell'} \\ &y(\ell, \bar{i}) \;=\; y(\ell', \bar{i}') \quad \text{implies} \quad L(\ell)|_{\bar{i}} \;=\; L(\ell')|_{\bar{i}'} \end{aligned} \tag{$\star$}$$

We recall that $T$ is a decomposition of a dense order $\mathbb{D}$, and hence $Y_0 \subseteq Y_1 \subseteq \ldots \subseteq \mathbb{D}$. We can thus associate with each pair $(\ell, \ell') \in \mathbb{N}^2$, with $\ell \le \ell'$, an injective function $g_{\ell,\ell'}$ from $I_\ell$ to $I_{\ell'}$ in such a way that

$$\forall \ell \le \ell' \quad \forall i \in I_\ell \quad \forall i' \in I_{\ell'} \qquad g_{\ell,\ell'}(i) \;=\; i' \quad \text{iff} \quad y_{\ell,i} \;=\; y_{\ell',i'} \tag{$\dagger$}$$

Note that the above functions are *additive*, namely, $g_{\ell,\ell''} = g_{\ell',\ell''} \circ g_{\ell,\ell'}$, for all $\ell \le \ell' \le \ell''$. By exploiting the additivity of the functions $g_{\ell,\ell'}$ and a simple induction on $|\ell' - \ell|$, one can easily verify that

$$L(\ell) \;=\; L(\ell')\big|_{g^{-1}_{\ell,\ell'}(I_{\ell'})}. \tag{$\ddagger$}$$

Finally, the desired property ($\star$) follows from ($\dagger$) and ($\ddagger$) by observing that

$$g_{\ell,\ell'}(\bar{i}) \;=\; \bar{i}' \qquad \text{implies} \qquad L(\ell)|_{\bar{i}} \;=\; \Big(L(\ell')\big|_{g^{-1}_{\ell,\ell'}(I_{\ell'})}\Big)\Big|_{\bar{i}} \;=\; L(\ell')|_{\bar{i}'}$$

□

We are finally ready to translate full profile trees to globally fulfilling compass structures.

*Proof of 2nd statement of Proposition 2.* Let $\mathcal{T} = (T, N, E)$ be a full profile tree, where $T$ is a decomposition of some dense order with endpoints $\mathbb{D}$. We first define the domain of the compass structure $\mathcal{G}$ that corresponds to $\mathcal{T}$. This is the subset of $\mathbb{D}$ that consists of all time points appearing in the nodes of $T$, that is, the set $\mathbb{D}' = \bigcup_{(y_1,y_2)\in T}\{y_1, y_2\}$. Clearly, $\mathbb{D}'$ is a dense sub-order of $\mathbb{D}$ with endpoints.

To define the labelling $\tau$ of $\mathcal{G}$, we exploit Lemma 5. First, we construct the function $L$ mapping natural numbers $\ell$ to multisets $L(\ell)$ containing $(n_\ell + 1)$-tuples of atoms, where $n_\ell$ is the number of nodes of $T$ at level $\ell$. For every $\ell \in \mathbb{N}$ and each $1 \le i \le n_\ell$, let $s_{\ell,i} = (y_{\ell,i}, y_{\ell,i+1})$ be the $i$-th node from the left at level $\ell$. By Lemma 5, it holds that:

- $N(s_{0,1}) = L(0)$ and $N(s_{\ell,i}) \subseteq L(\ell)|_{i,i+1}$ for all $1 \le i \le n_\ell$;
- $E(s_{\ell,i}) \subseteq L(\ell+1)|_{\{j,\ldots,j+n\}}$, with $s_{\ell+1,j}$, …, $s_{\ell+1,j+n-1}$ children of $s_{\ell,i}$;
- $L(\ell)|_{\{i_1,\ldots,i_h\}} \;=\; L(\ell')|_{\{i'_1,\ldots,i'_h\}}$ for all $\ell' \ge \ell$, $1 \le i_1 < \ldots < i_h \le n_\ell + 1$, and $1 \le i'_1 < \ldots < i'_h \le n_{\ell'} + 1$, with $y_{\ell,i_1} = y_{\ell',i'_1}$, …, $y_{\ell,i_h} = y_{\ell',i'_h}$;
- $\mathsf{req}_{\bar{B}}(G_k) \;=\; \bigcup_{k<h\le n_\ell+1} \mathsf{obs}(G_h)$ for all $\pi$-tuples $(G_1, \ldots, G_{n_\ell+1}) \in L(\ell)$ and all $1 \le k \le n_\ell + 1$, with $G_k \neq \varnothing$;
- $\mathsf{req}_B(G_k) \;=\; \bigcup_{1\le h<k} \mathsf{obs}(G_h)$ for all $\pi$-tuples $(G_1, \ldots, G_{n_\ell+1}) \in L(\ell)$ and all $1 \le k \le n_\ell + 1$, with $G_k \neq \varnothing$.

Consider now a generic point $p = (x, y)$, with $x, y \in \mathbb{D}'$. We observe that the set of coordinates $Y_\ell = \{y_{\ell,i} \ : \ 1 \le i \le n_\ell + 1\}$ increases with $\ell$, that is, $Y_\ell \subseteq Y_{\ell+1}$, and, for sufficiently large $\ell$, this set covers any element of the domain $\mathbb{D}'$ (it holds that $\bigcup_{\ell\in\mathbb{N}} Y_\ell = \mathbb{D}'$). This means that there is a triple $(\ell, i, j)$, with $\ell \in \mathbb{N}$ and $1 \le i, j \le n_\ell + 1$, such that $x = y_{\ell,i}$ and $y = y_{\ell,j}$. The label $\tau(x, y)$ will be defined by identifying a suitable tuple in $L(\ell)$ on the basis of the index $i$ and by projecting it onto the $j$-element. As we will see, the definition does not depend on the particular choice of the triple $(\ell, i, j)$.

Given the triple $(\ell, i, j)$, we first show that there exists exactly one tuple $\bar{G} = (G_1, \ldots, G_{n_\ell+1})$ in $L(\ell)$ such that $G_i$ is a $\pi$-atom. By Definition 1, the profile $N(s_{\ell,i})$ contains a unique occurrence of a pair of the form $(F, G)$, with $F$ $\pi$-atom, and since $N(s_{\ell,i}) \sqsubseteq L(\ell)|_{i,i+1}$, this pair $(F, G)$ must be the projection onto the components $G_i, G_{i+1}$ of a unique tuple $(G_1, \ldots, G_{n_\ell+1})$ in $L(\ell)$. Intuitively, the atom $G_i$ of the distinguished tuple $\bar{G} = (G_1, \ldots, G_{n_\ell+1})$ represents the atom that must be associated with the point $(x, x)$; similarly, the atom $G_j$ represents the atom that must be associated with the point $(x, y)$ (note that $G_j = \varnothing$ iff $j < i$ iff $y < x$). We thus define $\tau(x, y) = G_j$.

It is worth remarking that the above definition of $\tau(x, y)$ does not depend on the particular choice of the triple $(\ell, i, j)$, as choosing any other triple $(\ell', i', j')$ such that $x = y_{\ell',i'}$ and $y = y_{\ell',j'}$ would have led to the same definition of $\tau(x, y)$. This follows essentially from the fact that the definition depends only on the $i$-th and $j$-th components of the tuples in $L(\ell)$ and from the fact that

$$\begin{cases} y_{\ell,i} = x = y_{\ell',i'} \\ y_{\ell,j} = y = y_{\ell',j'} \end{cases} \qquad \text{implies} \qquad L(\ell)|_{i,j} \ = \ L(\ell')|_{i',j'}$$

Before proving that the resulting structure $\mathcal{G} = (\mathbb{D}' \times \mathbb{D}', \tau)$ is a globally fulfilling compass structure, we show that for all nodes $s_{\ell,i}$, we have $\mathsf{set}(\mathsf{profile}(\mathcal{G}_{s_{\ell,i}})) = \mathsf{set}(N(s_{\ell,i}))$. Consider a pair $(F, G)$ that occurs in the profile of a slice $\mathcal{G}_{s_{\ell,i}}$. Clearly, there exist two points $p = (x, y_{\ell,i})$ and $q = (x, y_{\ell,i+1})$ such that $\tau(p) = F$ and $\tau(q) = G$. By construction, there exists a quadruple $(\ell', i', j', k')$ such that $x = y_{\ell',i'}$, $y_{\ell',j'} = y_{\ell,i}$, and $y_{\ell',k'} = y_{\ell,i+1}$, and hence there exists a tuple $(G_1, \ldots, G_{n_{\ell'}+1})$ in $L(\ell')$ such that $G_{i'}$ is a $\pi$-atom, $G_{j'} = \tau(p) = F$, and $G_{k'} = \tau(q) = G$. We know from the properties of $L$ that $L(\ell')|_{j',k'} = L(\ell)|_{i,i+1} \sqsupseteq N(s_{\ell,i})$. This means that the pair $(F, G)$ occurs also in the profile $N(s_{\ell,i})$, possibly with a different multiplicity. The converse direction follows by symmetric arguments.

We first prove that $\mathcal{G} = (\mathbb{D}' \times \mathbb{D}', \tau)$ is a *consistent* compass structure. Consider two points $p, q$ in $\mathcal{G}$ such that $p \; \bar{B} \; q$, say $p = (x, y)$ and $q = (x, z)$, with $x, y, z \in \mathbb{D}'$ and $x \le y < z$. We know that, sufficiently down in the tree $T$, say at some level $\ell$, we can find a sequence of pairwise adjacent nodes $s_{\ell,i} = (y_{\ell,i}, y_{\ell,i+1})$, ..., $s_{\ell,j} = (y_{\ell,j}, y_{\ell,j+1})$, ..., $s_{\ell,k} = (y_{\ell,k}, y_{\ell,k+1})$, with $1 \le i \le j < k \le n_\ell + 1$, such that $y_{\ell,i} = x$, $y_{\ell,j} = y$, and $y_{\ell,k} = z$. Let $(G_1, \ldots, G_{n_\ell+1})$ be the unique tuple in $L(\ell)$ such that $G_i$ is a $\pi$-atom. By construction, we have that $\tau(x, x) = G_i$, $\tau(p) = G_j$, and $\tau(q) = G_k$. We prove by induction on $k - j$ that $\tau(p) = G_j \uparrow G_k = \tau(q)$ (see Section 2 for the definition of the relation $\uparrow$). The base case $k - j = 1$ holds trivially: it immediately follows from $(G_j, G_k) \in L(\ell)|_{j,k} \sqsupseteq N(s_{\ell,j})$ and the first condition of Definition 1. The induction step is straightforward as well, since, by transitivity of $\uparrow$, it reduces to the application of the inductive hypothesis: if $k - j > 1$, then $G_j \uparrow G_k$ follows from $G_j \uparrow G_{j+1} \uparrow G_k$.

Consider now two points $p, q$ in $\mathcal{G}$ such that $p\ A\ q$, namely, $p = (x, y)$ and $q = (y, z)$, for some $x, y, z \in \mathbb{D}'$ with $x \leq y \leq z$. As before, at some level $\ell$ in $T$, we can find a sequence of pairwise adjacent nodes $s_{\ell,i} = (y_{\ell,i}, y_{\ell,i+1})$, ..., $s_{\ell,j} = (y_{\ell,j}, y_{\ell,j+1})$, ..., $s_{\ell,k} = (y_{\ell,k}, y_{\ell,k+1})$, with $1 \leq i \leq j \leq k \leq n_\ell + 1$, such that $y_{\ell,i} = x$, $y_{\ell,j} = y$, and $y_{\ell,k} = z$. Let $(G_1, \ldots, G_{n_\ell+1})$ be the unique tuple in $L(\ell)$ such that $G_i$ is a $\pi$-atom. Similarly, let $(H_1, \ldots, H_{n_\ell+1})$ be another unique tuple in $L(\ell)$ such that $H_j$ is a $\pi$-atom. By construction, $\tau(x, x) = G_i$, $\tau(p) = G_j$, $\tau(y, y) = H_j$, and $\tau(q) = H_k$. Since both pairs $(G_j, G_{j+1})$ and $(H_j, H_{j+1})$ occur in $L(\ell)|_{j,j+1} \sqsupseteq N(s_{\ell,j})$ and $N(s_{\ell,j})$ satisfies the second condition of Definition 1, it follows that $\mathsf{req}_A(G_j) = \mathsf{req}_A(H_j)$. Furthermore, since $H_j$ is a $\pi$-atom, by the definition of atom, it holds $\mathsf{req}_A(H_j) = \mathsf{obs}(H_j) \cup \mathsf{req}_{\bar{B}}(H_j)$. In addition, from the first condition of Definition 1, we have $H_j \uparrow H_k$. Putting all together, we obtain $\tau(p) = G_j \rightharpoonup H_j \uparrow H_k = \tau(q)$, which finally proves $\tau(p) \rightharpoonup \tau(q)$.

We now prove that $\mathcal{G} = (\mathbb{D}' \times \mathbb{D}', \tau)$ is *globally fulfilling*. We first show that all the requests along the direction $\bar{B}$ are fulfilled. Consider a point $p = (x, y)$ in $\mathcal{G}$ and a request $\psi \in \mathsf{req}_{\bar{B}}(\tau(p))$ of it. As usual, let $(\ell, j, k)$ be a triple such that $x = y_{\ell,j}$ and $y = y_{\ell,k}$, and let $(G_1, \ldots, G_{n_\ell+1})$ be the unique tuple in $L(\ell)$ such that $G_j$ is a $\pi$-atom. By construction, $\tau(p) = G_k$. Moreover, by the properties of $L$ that we stated at the beginning of the proof, $\mathsf{req}_{\bar{B}}(G_k) = \bigcup_{k<h\leq n_\ell+1} \mathsf{obs}(G_h)$, and thus $\psi \in \mathsf{obs}(G_h)$ for some $h > k$. By construction, the point $q = (x, y_{\ell,h})$ satisfies $p\ \bar{B}\ q$ and is labelled with the atom $G_h$, from which it follows $\psi \in G_h = \tau(q)$. Similar arguments can be used to prove that all requests of $p$ along the direction $B$ are fulfilled.

Next, we prove that all the requests along the direction $A$ are fulfilled. Consider a point $p = (x, y)$ and a request $\psi \in \mathsf{req}_A(\tau(p))$. As before, let $(\ell, j, k)$ be a triple such that $x = y_{\ell,j}$ and $y = y_{\ell,k}$, and let $(G_1, \ldots, G_{n_\ell+1})$ be the unique tuple in $L(\ell)$ such that $G_j$ is a $\pi$-atom. Clearly, $\tau(p) = G_k$. By definition of $L$, $N(s_{\ell,k}) \sqsubseteq L(\ell)|_{k,k+1}$. Let us assume $N_\pi(s_{\ell,k}) = (F, G)$ (if $k$ is equal to $n_\ell + 1$, then a similar argument holds by substituting $k - 1, k$ for $k, k + 1$ and $N^\pi(s_{\ell,k-1})$ for $N_\pi(s_{\ell,k})$). By the second and the fourth conditions of Definition 1, $\psi \in \mathsf{req}_A(G_k) = \mathsf{req}_A(F)$ and $F$ is a $\pi$-atom. It follows that $\mathsf{req}_A(F) = \mathsf{obs}(F) \cup \mathsf{req}_{\bar{B}}(F)$ and thus $\psi$ is an observable of the point $q = (y, y)$ or a request of it along the direction $\bar{B}$. In the first case, it immediately follows that $p\ A\ q$; in the second case, by the previous arguments, we can conclude that the request $\psi$ of $q$ along the direction $\bar{B}$ is fulfilled by a third point $r = (y, z)$, with $p\ A\ q\ \bar{B}\ r$.

Finally, consider a request $\psi \in \mathsf{req}_{\bar{A}}(\tau(p))$ of a point $p = (y, z)$. Let $(\ell, j, k)$ be a triple such that $y = y_{\ell,j}$ and $z = y_{\ell,k}$ and let $(G_1, \ldots, G_{n_\ell+1})$ be the unique tuple in $L(\ell)$ such that $G_j$ is a $\pi$-atom. By the previous results, it holds that $G_j \uparrow G_k = \tau(p)$, and thus $\mathsf{req}_{\bar{A}}(G_j) = \mathsf{req}_{\bar{A}}(G_k)$ (see Section 2 for the definition of $\uparrow$). We claim that $j > 1$. By contradiction, if $j = 1$, then $(G_j, G_{j+1}) = N_\pi(s_{\ell,1})$, as $\mathcal{T}$ is a *full* profile tree and $G_j$ is a $\pi$-atom. Moreover, since the profile $N(s_{\ell,1})$

contains only pairs $(F,G)$ with either $F = \varnothing$ or $\mathsf{req}_{\bar{A}}(F) = \varnothing$, by the last condition of Definition 1, it holds that $\mathsf{req}_{\bar{A}}(G_j) = \varnothing$, which contradicts $\psi \in \mathsf{req}_{\bar{A}}(G_k) = \mathsf{req}_{\bar{A}}(G_j)$. From $j > 1$ and the fact that $G_j$ is a $\pi$-atom, it immediately follows that $G_{j-1} = \varnothing$. Consider now the pair $(G_{j-1}, G_j) = (\varnothing, G_j)$, which belongs to the profile $N(s_{\ell,j-1}) \sqsubseteq L(\ell)|_{j-1,j}$. From the last condition of Definition 1, we get $\mathsf{req}_{\bar{A}}(G_j) = \bigcup_{(F,G)\in N(s_{\ell,j})} \mathsf{obs}(F)$, and hence $\psi \in \mathsf{obs}(G)$ for some pair $(F,G) \in N(s_{\ell,j-1}) \sqsubseteq L(\ell)|_{j-1,j}$. By condition **(profile-dummy)** of Definition 2, there exists a tuple $(F_1, \ldots, F_{n+1}) \in E(s_{\ell,j-1})$ such that $F_{n+1} = G$ and $F_i$ is a $\pi$-atom for some $i \leq n+1$. Moreover, it holds that $E(s_{\ell,j-1}) \sqsubseteq L(\ell+1)|_{\{k,\ldots,k+n\}}$, for some index $k$, and hence $(F_i, G) \in L(\ell+1)|_{k+i-1,k+n}$. Finally, we recall that $y_{\ell+1,k+n} = y_{\ell,j} = y$ and hence the point $q = (x,y)$, where $x = y_{\ell+1,k+i-1}$, satisfies $p = (y,z)\ \bar{A}\ q$ and is labelled with the atom $G$. This proves that the request $\psi$ of $p$ is fulfilled at $q$. □

**Proposition 3.** *For every finite pseudo-regular profile tree $\mathcal{T}$, there is an infinite profile tree $\mathcal{T}'$ that has the same profile as $\mathcal{T}$ at the root.*

*Proof.* The proof of this result is not very difficult. Without loss of generality, one can assume that the finite pseudo-regular tree $\mathcal{T} = (T, N, E)$ is minimal with respect to the prefix partial order. This implies that every leaf $s'$ of $\mathcal{T}$ has a proper ancestor $s$ such that $N(s) \sqsubseteq N(s')$ and $N(s)(\varnothing, G) = N(s')(\varnothing, G)$ for all atoms $G$. One then uses the latter property as an invariant of a construction that repeatedly extends the frontier of $\mathcal{T}$ by a new set of leaves, each one associated with a profile that is roughly obtained as in an arithmetic progression. The limit of such a construction gives an infinite pseudo-regular profile tree $\mathcal{T}'$ that extends $\mathcal{T}$.

Let $\mathcal{T} = (T, N, E)$ be a finite pseudo-regular profile tree. W.l.o.g., we can assume $\mathcal{T}$ to be minimal with respect to the prefix partial order (if this is not the case, then a minimal prefix of $\mathcal{T}$ can be obtained by pruning the sub-trees strictly below the nodes $s'$ that have proper ancestors $s$ in $\mathcal{T}$ satisfying $N(s) \sqsubseteq N(s')$ and $N(s)(\varnothing, G) = N(s')(\varnothing, G)$ for all atoms $G$). The infinite profile tree $\mathcal{T}'$ can be obtained as the limit of a construction that repeatedly extends the frontier of $\mathcal{T}$ with new leaves. More precisely, we inductively build a series of finite profiles trees $\mathcal{T}_i = (T_i, N_i, E_i)$ such that, for all $\geq 0$:

1. $T_i \subsetneq T_{i+1}$;
2. $N_{i+1}(s) = N_i(s)$, for all nodes $s \in T_i$;
3. $E_{i+1}(s) = E_i(s)$, for all internal nodes $s \in T_i \setminus \mathsf{fr}(T_i)$, where $\mathsf{fr}(T_i)$ denotes the set of leaves of $T_i$;
4. $\mathsf{fr}(T_i) \cap \mathsf{fr}(T_j) = \varnothing$ for some $j > i$ (this guarantees that any leaf of $T_i$ eventually becomes an internal node in some $T_j$).

To do this, we will exploit the following invariant:

$$\forall s' \in \mathsf{fr}(T_i)\quad \exists s \in T_i \text{ proper ancestor of } s' \quad \begin{cases} N_i(s) \sqsubseteq N_i(s') \\ N_i(s)(\varnothing, G) = N_i(s')(\varnothing, G) \\ \text{for all atoms } G. \end{cases} \qquad (\star)$$

As for the base step, we let $\mathcal{T}_0 = (T_0, N_0, E_0) = \mathcal{T}$. The invariant ($\star$) holds for $\mathcal{T}$ by definition of pseudo-regular tree.

As for the inductive step, suppose that some $\mathcal{T}_i = (T_i, N_i, E_i)$ satisfying ($\star$) is defined and fairly choose a leaf $s' \in \mathsf{fr}(T_i)$ (here fairly means that any element of $\mathsf{fr}(T_i)$ is eventually chosen). Thanks to the invariant ($\star$), we can find a proper ancestor $s$ of $s'$ such that $N_i(s) \sqsubseteq N_i(s')$ and $N_i(s)(\varnothing, G) = N_i(s')(\varnothing, G)$, for all atoms $G$. We construct $\mathcal{T}_{i+1}$ by adding new nodes under the leaf $s'$ of $\mathcal{T}_i$ and by properly replicating the labelling structure of $s$ and its children on $s'$ and its children. More precisely, let $s_1, \ldots, s_n$ be the children of $s$. We accordingly add new nodes $s'_1, \ldots, s'_n$ as children of $s'$ in $T_{i+1}$, that is, $T_{i+1} = T_i \uplus \{s'_1, \ldots, s'_n\}$. For all nodes $s'' \in T_i$, we simply let $N_{i+1}(s'') = N_i(s'')$; similarly, for all nodes $s'' \in T_i \setminus \{s'\}$, we let $E_{i+1}(s'') = E_i(s'')$. To define the multiset $E_{i+1}(s')$ and the profiles of the children of $s'$ in $\mathcal{T}_{i+1}$, we begin by setting $D = N_i(s') \setminus N_i(s)$, namely, we let $D$ be the multiset of pairs of atoms such that $D(F,G) = N_i(s')(F,G) - N_i(s)(F,G)$, for all $F, G$. Then, for each pair $(F,G)$ in the support of $D$, we fix a corresponding $(n+1)$-tuple of atoms $\bar{H}_{F,G}$ of the form $(F_1, H_2, \ldots, H_n, G)$, for some atoms $H_2, \ldots, H_n$, in the support of $E_i(s)$. We observe that (i) such a tuple $\bar{H}_{F,G}$ exists since $E_i(s)|_{1,n+1} = N_i(s)$ (cf. condition **(profile-match)** of Definition 2) and (ii) the tuple $\bar{H}_{F,G}$ is *not* a $\pi$-tuple (otherwise, by definition of $\sqsubseteq$, we would have that $N_i(s')(F,G) = N_i(s)(F,G)$ and hence the pair $(F,G) \notin \mathsf{set}(D)$). In particular, distinct pairs $(F,G)$ are mapped to distinct tuples $\bar{H}_{F,G}$. We then define

$$E_{i+1}(s')(\bar{H}) \;=\; \begin{cases} E_i(s)(\bar{H}_{F,G}) + D(F,G) & \text{if } \bar{H} = \bar{H}_{F,G} \text{ for some } (F,G) \in D; \\ E_i(s)(\bar{H}) & \text{otherwise.} \end{cases}$$

Accordingly, we let $N_{i+1}(s'_j) = E_{i+1}(s')|_{j,j+1}$ for each $j = 1, \ldots, n$. By construction, we have that $E_{i+1}(s')|_{1,n+1} = N_i(s) \cup D = N_i(s') = N_{i+1}(s')$ and $E_{i+1}(s')|_{j,j+1} = N_{i+1}(s'_j)$, and thus $\mathcal{T}_{i+1} = (T_{i+1}, N_{i+1}, E_{i+1})$ satisfies the matching conditions of Definition 2. Hence, $\mathcal{T}_{i+1}$ is a profile tree. To complete the proof, it remains to check that the invariant ($\star$) holds on the new leaves of $\mathcal{T}_{i+1}$. This is, however, easy to do since, for each leaf $s'_j$ of $s'$, it holds that:

1. $N_{i+1}(s'_j) = E_{i+1}(s')|_{j,j+1} \subseteq E_i(s')|_{j,j+1} = N_i(s_j)$;
2. $\mathsf{set}\big(N_{i+1}(s'_j)\big) = \mathsf{set}\big(E_{i+1}(s')|_{j,j+1}\big) = \mathsf{set}\big(E_i(s')|_{j,j+1}\big) = \mathsf{set}\big(N_i(s_j)\big)$, because any pair $(F,G)$ that belongs to the support of $D$ ($= N_i(s') \setminus N_i(s)$) must also belong to the support of $N_i(s)$ (otherwise $N_i(s) \not\sqsubseteq N_i(s')$), and hence $\mathsf{set}\big(E_{i+1}(s')\big) = \mathsf{set}\big(E_i(s)\big)$;
3. $N_{i+1}(s'_j)(\varnothing, G) = N_i(s_j)(\varnothing, G)$ for all atoms $G$, because $E_{i+1}(\bar{H}) = E_i(\bar{H})$ for all $\pi$-tuples $\bar{H}$.

The limit of the above construction, defined as $\mathcal{T}' = (T', N', E')$, where $T' = \bigcup_{i \in \mathbb{N}} T_i$, $N'(s) = N_i(s)$, and $E'(s) = E_i(s)$ for large enough $i$ such that $s \in T_i$, is an infinite profile tree that extends $\mathcal{T}$. In particular, $\mathcal{T}'$ has the same profile as $\mathcal{T}$ at the root. $\square$

**Lemma 1.** *If $N$ is a feasible profile and $N'$ is a profile such that $N \sqsubseteq N'$ and $N(\varnothing, G) = N'(\varnothing, G)$ for all atoms $G$, then $N'$ is feasible too. Moreover, a profile* *tree with root profile $N'$ can be obtained from a profile tree with root profile $N$ without modifying the underlying decomposition tree.*

*Proof.* The proof is by induction on the number of pairs $(F,G)$, with $F \neq \emptyset$, such that $N(F,G) < N'(F,G)$ (the base case being trivial). Each inductive step is based a suitable transformation performed on the profile tree $\mathcal{T}$ that witnesses the feasibility of $N$. Intuitively, the transformation increases the multiplicity of a specific pair $(F,G)$ in the root profile $N$ and then propagates the inflation to the children, using the matching conditions of Definition 2. We remark that this transformation is possible (and easy) because it does not affect the multiplicities of the pairs of the form $(\emptyset,G)$, which usually encode more complex constraints.

Let us consider a pair $(F,G)$, with $F \neq \emptyset$, such that $N(F,G) < N'(F,G)$. Since $N$ is feasible, there is a profile tree $\mathcal{T} = (T,\tilde{N},E)$, with root $r$, such that $\tilde{N}(r) = N$. In the following, we apply to $\mathcal{T}$ a suitable transformation that results in a new profile tree $\mathcal{T}' = (T,\tilde{N}',E')$ that has the same decomposition tree as $\mathcal{T}$ and satisfies (i) $\tilde{N}(s) \sqsubseteq \tilde{N}'(s)$, for all $s \in T$, (ii) $\tilde{N}'(r) \sqsubseteq N'$, and (iii) $\tilde{N}'(r)(F,G) = N'(F,G)$. The transformation is obtained by increasing the multiplicities in $\tilde{N}(r)$ and in $E(r)$ and by recursively transforming the subtrees rooted at the children of $r$ so as to respect the conditions of Definition 2 as follows.

- The profile $\tilde{N}'(r)$ is defined by $\tilde{N}'(r)(F,G) = N'(F,G)$ and $\tilde{N}'(r)(F',G') = N(F',G')$ for all pairs $(F',G') \neq (F,G)$.
- Let $s_1,\dots,s_n$ be the children of $r$. We know from condition **(profile-match)** of Definition 2 that the multiset $E(r)$ contains at least one tuple of the form $\bar{F} = (F_1,\dots,F_{n+1})$, with $F_1 = F$ and $F_{n+1} = G$. We fix any such tuple $\bar{F}$ and we define
$$E'(r)(\bar{F}) \;=\; E(r)(\bar{F}) \;+\; N'(F,G) \;-\; N(F,G).$$
For all other tuples $\bar{F}' \in E(r)$, with $\bar{F}' \neq \bar{F}$, we define $E'(r)(\bar{F}') = E(r)(\bar{F}')$.
- The labellings for the subtrees of $\mathcal{T}'$ under $r$ are defined as follows. First, for each index $1 \le i \le n$, we define $\tilde{N}'(s_i) = E'(r)|_{i,i+1}$. It can be easily checked that $\tilde{N}(s_i) \sqsubseteq \tilde{N}'(s_i)$ and $\tilde{N}(s_i)(\emptyset,G') = E'(r)|_{i,i+1}(\emptyset,G')$ for all atoms $G'$. Thus, we can apply inductively the transformation to each subtree rooted at $s_i$ using the profile $\tilde{N}'(s_i)$ as an inflation of $\tilde{N}(s_i)$.

The result of the above inductively-defined transformation is a profile tree $\mathcal{T}' = (T,\tilde{N}',E')$ such that $\tilde{N}(s) \sqsubseteq \tilde{N}'(s)$, for all $s \in T$, (ii) $\tilde{N}'(r) \sqsubseteq N'$, and (iii) $\tilde{N}'(r)(F,G) = N'(F,G)$.

Finally, by iterating the above transformation on all pairs $(F,G)$ such that $F \neq \emptyset$ and $N(F,G) < N'(F,G)$, we obtain the desired profile tree witnessing the feasibility of $N'$. ☐

**Lemma 2.** *For every feasible profile $N$, there is an infinite pointwise fair profile tree that has root profile $N' \sqsupseteq N$. Moreover, one can assume that, for all pairs of atoms $(F,G)$, if $N|_1(F) < \infty$, then $N(F,G) = N'(F,G)$.*

The basic principle of the proof of the above lemma is similar to that of Lemma 1, since we use an inductively-defined transformation on infinite profile trees. However, differently from the previous proof, here we correct several violations of fairness at once, and *level-wise*. In particular, in this proof we make an extensive use of Lemma 5. Before entering the details of the proof, however, we make some preliminary remarks and introduce further terminology and definitions.

We recall that, given any profile tree $\mathcal{T} = (T, N, E)$, Lemma 5 produces a function $L$ that maps any level $\ell \in \mathbb{N}$ to a multiset $L(\ell)$ of $(n_\ell + 1)$-tuples of atoms satisfying some desired properties (i.e., the five items of the claim of the lemma). For shortness, we say that $L$ is the *level function induced by* $\mathcal{T}$. A converse construction also holds that takes a level function $L$ satisfying all the properties of Lemma 5 but the first two (which are irrelevant now since the profile tree is unspecified), and produces an infinite profile tree $\mathcal{T}_L$. The construction of $\mathcal{T}_L$ is straightforward:

- the decomposition tree $T_L$ of $\mathcal{T}_L$ is any decomposition of a temporal domain that is compatible with the function $L$, namely, has exactly $n_\ell$ nodes at level $\ell$, where $n_\ell + 1$ is the length of the tuples in $L(\ell)$);
- the profile function $N_L$ of $\mathcal{T}_L$ is defined by $N_L(s_{\ell,j}) = L(\ell)|_{j,j+1}$, for all $\ell \in \mathbb{N}$ and all $1 \leq j \leq n_\ell$, where $s_{\ell,j}$ is the $j$-th node at level $\ell$ in $T_L$;
- similarly, the function $E_L$ is defined by $E_L(s_{\ell,j}) = L(\ell+1)|_{k,\ldots,k+n}$, for all $\ell \in \mathbb{N}$ and all $1 \leq j \leq n_\ell$, where $s_{\ell,j}$ is the $j$-th node at level $\ell$ and $s_{\ell+1,k}, \ldots, s_{\ell+1,k+n}$ are the children of $s_{\ell,j}$ in $T_L$.

We call $\mathcal{T}_L$ the profile tree *generated by a level function* $L$. Moreover, one easily verifies that, if $L$ is the level function induced by a profile tree $\mathcal{T}$ with the same decomposition tree as $\mathcal{T}_l$, then $\mathcal{T}_L$ "dominates" $\mathcal{T}$ on all nodes, namely, $\mathcal{T}_L(s) \sqsupseteq \mathcal{T}(s)$ for all $s \in T_L$. On the basis of this last remark, hereafter we can restrict ourselves to profile trees like $\mathcal{T}_L$ and identify them with the induced level functions $L$.

To ease an inductive proof of Lemma 2, we generalize the notion of fairness (cf. Definition 4) to the multisets specified by an induced level function $L$.

**Definition 6.** Let $L$ be a function mapping any level $\ell \in \mathbb{N}$ to a multiset of $(n_\ell + 1)$-tuples. We say that $L$ is *fair up to level* $\ell$ if for all $\ell' \leq \ell$ and all $(n_{\ell'} + 1)$-tuples $(\varnothing, \ldots, \varnothing, F_j, \ldots, F_{n_{\ell'}+1}) \in L(\ell')$, with $F_i \neq \varnothing$, we have that

$$L(\ell')|_{1,\ldots,j}(\varnothing, \ldots, \varnothing, F_i) = \infty \qquad \text{implies} \qquad L(\ell')(\varnothing, \ldots, \varnothing, F_j, \ldots, F_{n_{\ell'}+1}) = \infty.$$

*Proof of Lemma 2.* Let $\mathcal{T} = (T, N, E)$ be an infinite profile tree that has profile $N$ at the root. The basic principle of the proof of Lemma 2 is similar to that of Lemma 1, as we use an inductively-defined transformation on infinite profile trees. However, differently from the previous proof, here we correct several violations of fairness at once, and *level-wise*. In particular, in this proof we make an extensive use of Lemma 5.

The rough idea is to enforce the generalized fairness property over each level of the initial profile tree by increasing the multiplicities associated with some tuples in the induced level function and by correcting at the same time the possible violations of the matching conditions that could arise. The relabelling process produces a series of level functions $L_0, L_1, \ldots$ over the same decomposition tree $T$ such that, for every $\ell \in \mathbb{N}$, the function $L_\ell$ is fair up to level $\ell+1$ and $L_0(\ell') \sqsupseteq L_1(\ell') \sqsupseteq \ldots \sqsupseteq L_\ell(\ell')$ for all levels $\ell' \leq \ell+1$. We will then be able to define a "limit" of the level functions $L_0, L_1, \ldots$, which turns out to be fair up to any level, and from this we will finally obtain a pointwise fair profile tree $\mathcal{T}'$ with root profile $N' \sqsupseteq N$.

To simplify the notation, we will omit the subscripts $0, 1, \ldots$ form the level functions $L_0, L_1, \ldots$ and consider a generic step of the transformation that takes a level function $L$ (over the fixed decomposition tree $T$) that is fair up to level $\ell$ and returns a new level function $L'$ that is fair up to level $\ell+1$ and furthermore satisfies $L(\ell') \sqsubseteq L'(\ell')$ for all levels $\ell' \leq \ell+1$. We give the details of this transformation step below.

First of all, we define the set $\Gamma$ of tuples $(\varnothing, \ldots, \varnothing, F_j, \ldots, F_{n_{\ell+1}+1}) \in L(\ell+1)$ that violate the condition of Definition 6, namely, such that $F_j \neq \varnothing$, $L(\ell+1)|_{1,\ldots,j}(\varnothing, \ldots, \varnothing, F_j) = \infty$, and $L(\ell+1)(\varnothing, \ldots, \varnothing, F_j, \ldots, F_{n_{\ell+1}+1}) < \infty$. We observe the important fact that $\Gamma$ contains no $\pi$-tuples (otherwise, there would exist a profile in the tree generated from $L$ that violates Definition 1). We then define the function $L'$ on the $(\ell+1)$-th level in the obvious way:

$$L'(\ell+1)(\bar{F}) \quad = \quad \begin{cases} \infty & \text{if } \bar{F} \in \Gamma, \\ L(\ell+1)(\bar{F}) & \text{otherwise.} \end{cases}$$

Clearly, we have $L'(\ell+1) \sqsupseteq L(\ell+1)$.

To define $L'$ on the remaining levels, we must consider the equalities between the $y$-coordinates of nodes at different levels in the underlying decomposition tree $T$. For every $\ell' \in \mathbb{N}$ and every $1 \leq j' \leq n_{\ell'}+1$, we let $(y_{\ell',j'}, y_{\ell',j'+1})$ be the coordinates of the $j'$-th node at level $\ell'$ in $T$. Then, for the upper levels $\ell' = 0, \ldots, \ell$, we identify the set of positions at level $\ell+1$ that have the same $y$-coordinates as some positions at level $\ell'$, that is,

$$J^{\ell'}_{\underset{\ell+1}{\uparrow}} \quad = \quad \left\{ 1 \leq j \leq n_{\ell+1}+1 \ : \ y_{\ell+1,j} = y_{\ell',j'} \text{ for some } 1 \leq j' \leq n_{\ell'+1}+1 \right\}$$

and we accordingly define $L'(\ell') = L'(\ell+1)|_J$, where $J$ abbreviates $J^{\ell'}_{\underset{\ell+1}{\uparrow}}$ (note that again we have $L'(\ell') \ \sqsupseteq \ L(\ell+1)|_J \ = \ L(\ell'))$.

As concerns the lower levels $\ell' = \ell+2, \ell+3, \ldots$, we proceed as follows. We first observe that the function $L'$ that we defined so far satisfies all the properties of Lemma 5, but the first two, when relativised to the first $\ell+2$ levels. By applying standard constructions – that is, by projecting the multisets $L(\ell')$ over the appropriate components – we can generate from the partially defined function $L'$ a corresponding *finite* profile tree $\mathcal{T}'$. Formally, we let $\mathcal{T}' = (T', N', E')$, where

- $T'$ is the finite prefix of the decomposition tree $T$ that contains the first levels $0, \ldots, \ell+1$,
- $N'(s_{\ell',j}) = L(\ell')|_{j,j+1}$ for all $0 \leq \ell' \leq \ell+1$ and all $1 \leq j \leq n_{\ell'+1}$,
- $E'(s_{\ell',j}) = L(\ell'+1)|_{k,\ldots,k+n}$ for all $\ell' \leq \ell$ and all $1 \leq j \leq n_{\ell'+1}$, where $s_{\ell'+1,k}$, $\ldots$, $s_{\ell'+1,k+n}$ are the children of $s_{\ell',j}$ (note that the function $E'$ is undefined on the frontier of $T'$).

We then observe the following facts about the nodes $s_{\ell+1,j}$ at the frontier of $T'$:

1. $N(s_{\ell+1,j})$ is a feasible profile,
2. $N(s_{\ell+1,j}) \sqsubseteq N'(s_{\ell+1,j})$,
3. for all pairs of atoms $(\varnothing, G)$, there exists a tuple $\bar{F} = (\varnothing, \ldots, \varnothing, F_{j+1}, \ldots, F_{n_{\ell+1}+1})$ such that $F_{j+1} = G$ and $N(s_{\ell,j})(\varnothing, G) = \big(L(\ell+1)|_{j,j+1}\big)(\varnothing, F_{j+1}) = \big(L'(\ell+1)|_{j,j+1}\big)(\varnothing, F_{j+1}) = N'(s_{\ell+1,j})(\varnothing, G)$ (note that if $\bar{F} \in \Gamma$ and $L(\ell+1)(\bar{F}) < L'(\ell+1)(\bar{F}) = \infty$, the projection of $\bar{F}$ onto the first $j+1$ components has infinite multiplicity in $L(\ell+1)|_{1,\ldots,j+1}$ too).

We can thus apply Lemma 1 and claim that every profile $N'(s_{\ell+1,j})$ at the frontier of $\mathcal{T}'$ is *feasible*. In addition, we can relabel the subtrees of $\mathcal{T}$ at nodes $s_{\ell+1,1}, \ldots, s_{\ell+1,n_{\ell+1}}$ and obtain in this way some profile trees $\mathcal{T}'_1, \ldots, \mathcal{T}'_{n_{\ell+1}}$ that can be attached over the leaves of $\mathcal{T}'$ so as to complete it into a infinite profile tree. We can finally prolong the function $L'$ to the remaining levels $\ell+2, \ell+3, \ldots$ using the completion of $\mathcal{T}'$ and Lemma 5 – here we can assume, without loss of generality, that the level function induced by $\mathcal{T}'$ coincides with $L'$ on the levels $0, \ldots, \ell+1$ (if this were not the case, we could just update $L'$ with the new induced function by observing that doing so can only increase the multiplicities).

We have just finished our description of a single transformation step, which takes a level function $L$, fair up to level $\ell$, and returns a new level function $L'$ that is fair up to level $\ell+1$. By construction the transformation preserves the underlying decomposition tree $T$ and furthermore guarantees that $L(\ell') \sqsubseteq L'(\ell')$ for all $\ell' \leq \ell+1$. These properties enable the definition of the "limit" $L^\star$ of the resulting series of functions $L_0, L_1, \ldots$, as follows:

$$L^\star(\ell) \;=\; \sup\nolimits_{i \geq \ell} L_i(\ell)$$

(note that this is well-defined because the length of the tuples of the multisets $L_i(\ell)$ is constant). To conclude, we observe that $L^\star$ is fair up to any level and hence generates an infinite pointwise fair profile tree $\mathcal{T}_{L^\star}$ with root profile $N' \sqsupseteq N$.

To prove the claim of the last sentence of the lemma, we recall the definition of the level function $L'$ on the basis of level function $L$, where the latter was assumed to be fair up to level $\ell$. We observe that the two profiles $L(0)$ and $L'(0)$ may differ only by the multiplicities associated with the pairs of the form $\bar{F}|_{1,n_{\ell+1}+1}$, for some tuples $\bar{F} \in \Gamma$. Moreover, by construction, every tuple $\bar{F} = (\varnothing, \ldots, \varnothing, F_j, \ldots, F_{n_{\ell+1}+1})$ from $\Gamma$ satisfies $\big(L(\ell+1)|_{1,\ldots,j}\big)(\varnothing, \ldots, \varnothing, F_j) = \infty$, and hence $L(0)|_1(\bar{F}|_1) = \infty$. By contraposition and by a simple induction we conclude that $N(F,G) = N'(F,G)$ whenever $N|_1(F) < \infty$. $\square$

**Lemma 3.** *Every sequence of feasible profiles $N_0 \sqsubseteq N_1 \sqsubseteq \ldots$ has a supremum $\sup_i N_i$, defined by $(\sup_i N_i)(F,G) = \sup_{i \in \mathbb{N}} \big(N_i(F,G)\big)$ for all atoms $F, G$, that is a feasible profile.*

Concerning the above lemma, we remark that defining the supremum $\sup_i N_i$ of a sequence of profiles is trivial. The technicality of the above result lies in showing that $\sup_i N_i$ is feasible whenever $N_0, N_1, \ldots$ are. Intuitively, a profile tree $\mathcal{T}$ that witnesses the feasibility of $\sup_i N_i$ can be recursively constructed from some profile trees $\mathcal{T}_0, \mathcal{T}_1, \ldots$ witnessing the feasibility of $N_0, N_1, \ldots$. Special attention, however, must be paid in the presence of finite but unbounded multiplicities of the form $N_0(F,G)$, $N_1(F,G)$, ...: in this case, the condition **(profile-finite-req)** in the definition of profile tree implies that the numbers of children of the roots of $\mathcal{T}_0, \mathcal{T}_1, \ldots$ increase steadily. To enable an inductive construction of $\mathcal{T}$, one must be able to "aggregate" several consecutive children under a new single node, and accordingly lift this operation at the level of the profiles.

To prove Lemma 3, we establish a preliminary result that allows us to aggregate the profiles associated with a sequence of consecutive children of the root of some profile tree. Intuitively, this operation corresponds to the possibility of merging adjacent slices into a single larger slice.

**Lemma 6.** Let $\mathcal{T} = (T, N, E)$ be a profile tree, let $r$ be its root, and let $s_1, \ldots, s_n$ be the children of $r$, ordered from left to right. For every pair of indices $1 \le i \le j \le n$, one can find a profile tree $\mathcal{T}'$ whose root is labelled by the profile $E(r)|_{i,j+1}$.

*Proof.* The proof is a straightforward implication of Lemma 5. Let $(y_{\ell,j}, y_{\ell,j+1})$ denote the coordinates of the $j$-th node at level $\ell$ in the profile tree $\mathcal{T} = (T, N, E)$ and let $L$ be the function obtained by applying Lemma 5 to $\mathcal{T}$. We construct $\mathcal{T}' = (T', N', E')$ as follows:

- $T' = \{r\} \uplus \{s \in T \,:\, s \text{ descendant of } s_k \text{ in } T, \text{ for some } i \le k \le j\}$, where the node-to-child relationships are naturally inferred from those of $T$;
- $N'(r) = L(1)|_{i,j+1} = E(r)|_{i,j+1}$;
- $N'(s) = L(\ell')|_{i',j'+1}$, for all $s \in T' \setminus \{r\}$ and for some triple of indices $\ell', i', j'$ such that $(y_{\ell',i'}, y_{\ell',j'+1}) = s$ (note that, thanks to the properties satisfied by the function $L$, the multiset $L(\ell')|_{i',j'+1}$ is independent of the particular choice of such $\ell', i', j'$);
- $E'(r) = L(2)|_{i',\ldots,j'+1}$, where $i', j'$ are the unique indices such that $y_{2,i'} = y_{1,i}$ and $y_{2,j'+1} = y_{\ell,j+1}$ (note that this implies $E'(r)|_{1,n'} = N'(r)$, where $n' = j - i + 1$ is the number of children of $r$ in $T'$);
- $E'(s) = L(\ell')|_{i',\ldots,j'+1}$ for all $s \in T' \setminus \{r\}$, where $(y_{\ell',i'}, y_{\ell',i'+1})$, ..., $(y_{\ell',j'}, y_{\ell',j'+1})$ are the coordinates of the children of $s$ in $T'$.

It is routine to the verify that $\mathcal{T}'$ is a profile tree. □

*Proof of Lemma 3.* Let $N_0 \sqsubseteq N_1 \sqsubseteq \ldots$ be a chain of profiles. The definition of the supremum $N^\star = \sup_i N_i$ is trivial, as shown in the statement of the lemma. Below we assume that the profiles $N_0, N_1, \ldots$ occur at the roots of some profile trees $\mathcal{T}_0$, $\mathcal{T}_1$, ..., and we construct a new profile tree having the supremum profile $N^\star$ at its root.

We begin by identifying three noticeable sets of pairs of atoms on the basis of the series of multiplicities in the profiles $N_0, N_1, \ldots$:

$$P_{\max} = \big\{(F,G) \,:\, \exists i \in \mathbb{N}\ \forall j \ge i\ N_i(F,G) = N_j(F,G) < \infty\big\}$$
$$P_{\sup} = \big\{(F,G) \,:\, \forall i \in \mathbb{N}\ \exists j \ge i\ N_i(F,G) < N_j(F,G) < \infty\big\}$$
$$P_{\infty} = \big\{(F,G) \,:\, \exists i \in \mathbb{N}\ \forall j \ge i\ N_j(F,G) = \infty\big\}.$$

Clearly, given any pair of atoms $(F,G)$, depending on whether $(F,G) \in P_{\max}$, $(F,G) \in P_{\sup}$, or $(F,G) \in P_\infty$, we have $N^\star(F,G) = \max_i N_i(F,G)$, $N^\star(F,G) =$

$\infty$, or $N^\star(F,G) = \infty$, respectively. Without loss of generality (namely, by restricting to sub-sequences of profiles), we can further assume that the multiplicity of each pair $(F,G) \in P_\infty \cup P_{\max}$ in the profiles $N_0, N_1, \dots$ is constant (either finite or infinite).

Now, for every $i \in \mathbb{N}$, let $s_{i,1} = (y_{i,1}, y_{i,2})$, $s_{i,2} = (y_{i,2}, y_{i,3})$, $\dots$, $s_{i,n_i} = (y_{i,n_i}, y_{i,n_{i+1}})$ be the children of the root of $\mathcal{T}_i$, listed from left to right, and let $E_i$ be the multiset of $(n_i+1)$-tuples that is associated with the root of the profile tree $\mathcal{T}_i$. For each $i \in \mathbb{N}$, we choose a *minimal* subset $J_i = \{j_{i,1} < \dots < j_{i,k_i+1}\}$ of $\{1, \dots, n_i+1\}$ satisfying the following properties:

- $j_{i,1} = 1$ and $j_{i,k_i+1} = n_i + 1$;
- for every pair $(F,G) \in P_{\max}$, with $F \neq \varnothing$, $E_i|_{J_i}$ contains exactly $N^\star(F,G)$ ($= N_i(F,G)$ for any $i \in \mathbb{N}$) tuples of the form $(F_1, \dots, F_{k_i+1})$ such that $F_1 = F$, $F_{k_i+1} = G$, $\mathsf{req}_{\bar{B}}(F_1) = \bigcup_{2 \le j \le k_i+1} \mathsf{obs}(F_j) \cup \mathsf{req}_{\bar{B}}(F_{k_i+1})$, and $\mathsf{req}_B(F_{k_i+1}) = \bigcup_{1 \le j \le k_i} \mathsf{obs}(F_j) \cup \mathsf{req}_B(F_1)$;
- for every pair $(F,G) \in P_{\sup} \cup P_\infty$, with $F \neq \varnothing$, $E_i|_{J_i}$ contains at least one tuple of the form $(F_1, \dots, F_{k_i+1})$ such that $F_1 = F$, $F_{k_i+1} = G$, $\mathsf{req}_{\bar{B}}(F_1) = \bigcup_{2 \le j \le k_i+1} \mathsf{obs}(F_j) \cup \mathsf{req}_{\bar{B}}(F_{k_i+1})$, and $\mathsf{req}_B(F_{k_i+1}) = \bigcup_{1 \le j \le k_i} \mathsf{obs}(F_j) \cup \mathsf{req}_B(F_1)$;
- for every pair $(\varnothing, G) \in P_{\max} \cup P_{\sup} \cup P_\infty$, with $G \neq \varnothing$, $E_i|_{J_i}$ contains at least one $\pi$-tuple of the form $(F_1, \dots, F_{k_i+1})$ such that $F_{k_i+1} = G$..

It is worth noticing the resemblance between the last three conditions and the conditions **(profile-finite-req)**, **(profile-infinite-req)**, and **(profile-dummy)** in the definition of profile tree. This is not by chance, as the rationale underlying the above conditions is indeed to ease the construction of a profile tree witnessing the feasibility of $N^\star$. For the same reason, the pairs in $P_{\sup}$, which have finite unbounded multiplicities in $N_0, N_1, \dots$, are treated just like the pairs in $P_\infty$, as both appear with multiplicity $\infty$ in $N^\star$.

Since we required the set $J_i = \{j_{i,1} < \dots < j_{i,k_i+1}\}$ to be minimal, it is easy to see that its size is bounded over all $i \in \mathbb{N}$: indeed, each pair $(F,G)$ of atoms requires the existence of a *constant* number of tuples in $E_i|_{J_i}$, each one adding at most $|\varphi|$ indices to $J_i$. In particular, we can assume without loss of generality that the size of the sets $J_i$ is constant, say $|J_i| = k+1$, and hence $k_0 = k_1 = \dots = k$. The parameter $k$ is precisely the number of children that will added under the root of a profile tree in order to witness the feasibility of $N^\star$.

More precisely, the construction of a profile tree for $N^\star$ is done as follows. We first create a new node $r$ and associate the profile $N^\star$ with it. Then we apply Lemma 6 to each profile tree $\mathcal{T}_i$ and to each pair of nodes $s_{i,j_{i,h}}, s_{i,j_{i,h+1}}$, for $h = 1, \dots, k$, thus obtaining new profile trees $\mathcal{T}_{i,1}, \dots, \mathcal{T}_{i,k}$ witnessing the feasibility of the profiles $N_{i,1} = E_i|_{j_{i,1},\dots,j_{i,2}}$, $\dots$, $N_{i,k} = E_i|_{j_{i,k},\dots,j_{i,k+1}}$. Now, for each $h = 1, \dots, k$, we consider the series of profiles $(N_{i,h})_{i \in \mathbb{N}}$ and we denote by $N^\star_h$ its supremum, i.e., $N^\star_h = \sup_i N_{i,h}$. Similarly, we define $E^\star = \sup_i E_i|_{j_{i,1},\dots,j_{i,k+1}}$, and we observe that $N^\star_h = E^\star|_{h,h+1}$ for all $h = 1, \dots, k$. Finally, we associate the multiset $E^\star$ with $r$ and we attach $k$ subtrees under $r$ that are constructed by induction as if we had to prove feasibility of the limit profiles $N^\star_1, \dots, N^\star_k$. $\square$

**Corollary 1.** *For all feasible profiles $N$, there is a $\trianglelefteq$-maximal profile $N' \trianglerighteq N$.*

*Proof.* We observe that from any infinite $\trianglelefteq$-chain of feasible profiles, one can extract an infinite sub-sequence that is also a $\sqsubseteq$-chain. An immediate consequence of Lemma 3 is that every $\trianglelefteq$-chain has an *upper bound* (which, however, is not guaranteed to be a supremum in the partial order $\trianglelefteq$). In its turn, the existence of upper bounds on $\trianglelefteq$-chains implies the existence of feasible profiles that are *maximal* with respect to $\trianglelefteq$: this can be either seen as a consequence of Zorn's Lemma, or proved directly by way of contradiction using an straightforward induction on the countably many profiles. $\square$

**Proposition 4.** *For every infinite pointwise fair profile tree with root profile $N$, there is an infinite pointwise fair and pointwise $\trianglelefteq$-maximal profile tree with root profile $N' \trianglerighteq N$.*

*Proof.* Let $\mathcal{T} = (T, N, E)$ be an infinite pointwise fair profile tree that has profile $N$ at the root. The proof of this result is based again on an inductively-defined transformation of $\mathcal{T}$ that results in a new profile tree that is both pointwise fair and pointwise $\trianglelefteq$-maximal. The transformation is performed by a series of substitutions of profiles in $\mathcal{T}$. The rough idea consists of visiting $\mathcal{T}$ in breadth-first order while replacing every non $\trianglelefteq$-maximal profile with a dominating maximal one, which exists thanks to Corollary 1. However, as a result of a profile substitution one might obtain a tree that violates the matching conditions of Definition 2 (in particular, these violations may occur between the node with the replaced profile, the right siblings, and the parent of it). Luckily, these violations can be fixed by interleaving applications of Lemma 1 and Lemma 2.

Below we describe a single step of the transformation. For this we assume that $s$ is the first node of $\mathcal{T}$ in the breadth-first visit whose profile $N(s)$ is not $\trianglelefteq$-maximal. Using Corollary 1, we can find a new feasible profile $N'$ such that $N(s) \trianglelefteq N'$ and $N'$ is $\trianglelefteq$-maximal. Since $N'$ is feasible, we can apply Lemma 2 and obtain an infinite pointwise fair profile tree $\mathcal{T}'$ with root profile $N'' \sqsupseteq N'$.

We now prove that $N' \trianglelefteq N''$ holds, which in fact implies $N' = N''$ thanks to the $\trianglelefteq$-maximality of $N'$. Indeed, from $N' \sqsubseteq N''$ we immediately derive $N' \subseteq N''$ and $\mathsf{set}(N|_2) = \mathsf{set}(N'|_2)$. Moreover, given any pair of atoms $(F, G)$, with $F \neq \varnothing$, we can distinguish two cases: either $N|_1(F) = \infty$ or $N|_1(F) < \infty$. In the former case, we exploit the fairness of $\mathcal{T}$ to derive $N(F, G) = \infty$, whence $N''(F, G) \geq N'(F, G) \geq \infty = N(F, G)$. In the latter case, we first derive from $F \neq \varnothing$ and $N \trianglelefteq N'$ the fact that $N(F, H) = N'(F, H)$ holds for all atoms $H$, then we deduce $\infty > N|_1(F) = \sum_H N(F, H) = \sum_H N'(F, H) = N'|_1(F)$, and finally, by the second claim of Lemma 2, we conclude that $N'(F, G) = N''(F, G)$.

We have just shown that $\mathcal{T}'$ is an infinite pointwise fair profile tree that has the $\trianglelefteq$-maximal profile $N'$ at its root and, furthermore, $N(s) \trianglelefteq N'$. Intuitively, we would like to replace the subtree of $\mathcal{T}$ at node $s$ with the tree $\mathcal{T}'$. This is straightforward to do when $s$ is the root of the profile tree $\mathcal{T}$. However, when $s$ is not the root of $\mathcal{T}$, the substitution might result in a tree that violates the matching conditions of Definition 2. We temporarily denote by $\tilde{\mathcal{T}} = (\tilde{T}, \tilde{N}, \tilde{E})$ the tree that is obtained from the substitution of the subtree of $\mathcal{T}$ at node $s$

with $\mathcal{T}'$, and we show how to correct the possible violations of the matching conditions for $\tilde{\mathcal{T}}$.

Let $p$ denote the parent of $s$ and let $s_1, \dots, s_n$ be the children of $p$, listed from left to right. Without loss of generality, we can still denote by $s$ the node of $\tilde{\mathcal{T}}$ where the substitution occurred, so that $s = s_i$ for some index $i$ (namely, $s$ is the $i$-th children of $p$ in $\tilde{\mathcal{T}}$) and $s_{i+1}, \dots, s_n$ are the right siblings of $s$. We also denote by $\Gamma$ the set of all atoms $G$ such that $N(s)(\varnothing, G) < \tilde{N}(s)(\varnothing, G)$ – intuitively, these are the only atoms that are responsible for the differences between the multiplicities in $N(s)$ and the multiplicities in $\tilde{N}(s)$. We then observe a few facts that follow immediately from the definition of the partial order $\trianglelefteq$:

- $N(s) \subseteq \tilde{N}(s)$, but $N(s) \neq \tilde{N}(s)$ (recall that $\tilde{N}(s) = N'$ is $\trianglelefteq$-maximal, but $N(s)$ is not),
- $\mathsf{set}\big(N(s)|_2\big) = \mathsf{set}\big(\tilde{N}(s)|_2\big)$ (since $N(s) \trianglelefteq N' = \tilde{N}(s)$),
- $N(s)(F, G) = \tilde{N}(s)(F, G)$ for all atoms $F, G \neq \varnothing$,
- $\big(\tilde{N}(s)|_2\big)(G) > 0$, and hence $\big(N(s)|_2\big)(G) > 0$ for all atoms $G \in \Gamma$.

Now, for each atom $G \in \Gamma$, we fix arbitrarily a tuple $\bar{F}_G = (F_1, \dots, F_{n+1})$ that occurs in $\tilde{E}(p)$ and satisfies $F_{i+1} = G$ (note that this tuple exists since, by the matching conditions, the multiset $\tilde{E}(p)|_{i+1} = E(p)|_{i+1} = N(s)|_2$ contains at least one occurrence of the atom $G$). From this we construct the new tuple $\bar{F}_G^{\varnothing} = (\varnothing, \dots, \varnothing, F_{i+1}, \dots, F_{n+1})$ by replacing the first $i$ components of $\bar{F}_G$ with dummy atoms. Then, we define the multiset $D$ that consists of exactly $\tilde{N}(s)(G) - N(s)(G)$ occurrences of each tuple $\bar{F}_G^{\varnothing}$, for all $G \in \Gamma$.

To correct the violations in the tree $\tilde{\mathcal{T}}$, we begin by replacing the multiset $\tilde{E}(p)$ with the following union of multisets:

$$\tilde{E}'(p) \quad = \quad \tilde{E}(p) \ \cup \ D.$$

We observe that $\tilde{E}'(p)|_{1,\dots,i-1}$ may differ from $\tilde{E}(p)|_{1,\dots,i-1}$ only by the multiplicities of the dummy tuple $(\varnothing, \dots, \varnothing)$, which can be easily overlooked in the definition of profile tree (in fact, the dummy tuple occurs already in $\tilde{E}(p)|_{1,\dots,i-1}$ with multiplicity $\infty$). Similarly, we have $\tilde{E}'(p)|_{i,i+1} = \tilde{N}(s_i) \supsetneq N(s_i) = E(p)|_{i,i+1}$. Finally, for all $j = i+1, \dots, n+1$, we have that $\tilde{E}'(p)|_{j,j+1}$ differs from $E(p)|_{j,j+1}$ only by the For every infinite pointwise fair profile tree with root profile $N$, there exists an infinite pointwise fair and pointwise $\trianglelefteq$-maximal profile tree with root profile $N' \trianglerighteq N$. multiplicities of pairs of the form $(F, F')$, with $F, F' \neq \varnothing$. In particular, if we define $\tilde{N}'(s_j) = \tilde{E}'(p)|_{j,j+1}$ for every $j = i+1, \dots, n+1$, then we easily see that $\tilde{N}'(s_j) \sqsubseteq \tilde{N}(s_j)$ and $\tilde{N}'(s_j)(\varnothing, F') = \tilde{N}(s_j)(\varnothing, F')$ for all atoms $F'$. We can thus apply Lemma 1 and in succession Lemma 2, this way obtaining new profile trees $\tilde{\mathcal{T}}'_{i+1}, \dots, \tilde{\mathcal{T}}'_{n+1}$ that are pointwise fair and that can safely replace the nodes $s_{i+1}, \dots, s_{n+1}$ in $\tilde{\mathcal{T}}$.

Now, consider the tree $\mathcal{T}''$ that is obtained by selecting the subtree of $\tilde{\mathcal{T}}$ at note $p$ (the parent of $s$) and by redefining:

- the root profile to be $\tilde{N}'(p) = \tilde{E}'(p)|_{1,n+1}$,
- the profile of node $s$ (i.e., the $i$-th child of $p$) to be $\tilde{N}'(s) = \tilde{N}(s)$,
- the subtrees at the $j$-th child of $p$, for every $j = i+1, \dots, n+1$, to be $\tilde{\mathcal{T}}'_j$,.

It is clear that $\mathcal{T}''$ satisfies the matching conditions of the definition of profile tree, and hence the profile $\tilde{N}'(p)$ is feasible. It is also routine verifying that $N(p) \trianglelefteq \tilde{N}'(p)$. Indeed, we have:

1. $\tilde{N}'(p) = \tilde{E}'(p)|_{1,n+1} \supseteq \tilde{E}(p)|_{1,n+1} = \tilde{N}(p)$;
2. $\mathsf{set}\big(\tilde{N}'(p)|_2\big) = \mathsf{set}\big(\tilde{E}'(p)|_{n+1}\big) = \mathsf{set}\big(E(p)|_{n+1}\big) \cup \big\{\bar{F}^{\varnothing}_G|_{n+1} : G \in \Gamma\big\} = \mathsf{set}\big(E(p)|_{n+1}\big) \cup \big\{\bar{F}_G|_{n+1} : G \in \Gamma\big\} = \mathsf{set}\big(E(p)|_{n+1}\big) = \mathsf{set}\big(\tilde{N}'(p)\big)$ (note that $\bar{F}^{\varnothing}_G|_{n+1} = \bar{F}_G|_{n+1}$ and $\bar{F}_G \in E(p)$ for all $G \in \Gamma$);
3. for all atoms $F, G \neq \varnothing$, $\tilde{N}'(p)(F,G) = \big(\tilde{E}'(p)|_{1,n+1}\big)(F,G) = \big(\tilde{E}(p)|_{1,n+1} \cup D|_{1,n+1}\big)(F,G) = \big(\tilde{E}(p)|_{1,n+1}\big)(F,G) = \tilde{N}(p)(F,G)$ (note that $(D|_{1,n+1})(F, G) = 0$ whenever $F \neq \varnothing$).

Moreover, since $s$ was chosen to be the *first* node in the breadth-first visit of $\mathcal{T}$ having a non $\trianglelefteq$-maximal profile, and since $p$ precedes $s$ in this visit, we know that $N(p)$ is $\trianglelefteq$-maximal, and hence $N(p) = \tilde{N}'(p)$. This means that we can safely replace the subtree of $\mathcal{T}$ at $p$ with the profile tree $\mathcal{T}''$, thus obtaining a new profile tree $\mathcal{T}'''$. We finally observe that $\mathcal{T}'''$ coincides with $\mathcal{T}$ on all nodes that precede $s$ in the breadth-first visit, as well as on the descendants of the nodes that precede $s$ along the same level. In addition, the resulting profile tree $\mathcal{T}'''$ is pointwise fair and has a $\trianglelefteq$-maximal profile at node $s$.

One concludes the proof by repeatedly applying the above transformation until all profiles become $\trianglelefteq$-maximal (if the transformation does not terminate, then the desired tree could be defined as the limit of the resulting series of trees, which is well-defined). □

The last piece of the puzzle is the proof of Proposition 5.

**Proposition 5.** *Every infinite pointwise fair and pointwise $\trianglelefteq$-maximal profile tree is pseudo-regular.*

*Proof.* Let $\mathcal{T} = (T, N, E)$ be an infinite, pointwise fair and pointwise $\trianglelefteq$-maximal profile tree. By way of contradiction, suppose that $\mathcal{T}$ is not pseudo-regular. There exists an infinite path $\pi$ in $T$ such that, for all pairs of nodes $s, s' \in \pi$, with $s$ proper ancestor of $s'$, one of the following conditions holds:

1. $N(s) \not\sqsubseteq N(s')$,
2. or $N(s)(\varnothing, F) \neq N(s)(\varnothing, F)$ for some atom $F$.

Because the relation $\sqsubseteq$ on feasible profiles is a well partial order, condition 1 can hold only over finitely many pairs of nodes $s, s'$ in $\pi$. This means that we can extract from $\pi$ an infinite subsequence $\pi' = s_0, s_1, s_2, \ldots$ of nodes that are pairwise violating condition 1, namely, such that

$$N(s_0) \sqsubseteq N(s_1) \sqsubseteq N(s_2) \sqsubseteq \ldots .$$

In particular, every pair of nodes $s_i, s_j$ from $\pi'$ must satisfy condition 2. In addition, we can assume without loss of generality (that is, by further restricting the subsequence $\pi'$) that for each pair of atoms $(G, H)$ (possibly $G = \varnothing$), we have

1. either $N(s_0)(G,H) = N(s_1)(G,H) = N(s_2)(G,H) = \ldots$
2. or $N(s_0)(G,H) < N(s_1)(G,H) < N(s_2)(G,H) < \ldots$.

We call *constant pairs* the former type of pairs, for which the multiplicities remain constant, and *increasing pairs* the latter type of pairs, which have strictly increasing multiplicities.

We begin now to disclose a series of basic properties for the profiles along the infinite sequence of nodes $\pi' = s_0, s_1, s_2, \ldots$. Recall that all pairs of nodes in $\pi'$ satisfy condition 2; hence, there exists an *increasing* pair $(\varnothing, F^>)$:

$$N(s_0)(\varnothing, F^>) \ < \ N(s_1)(\varnothing, F^>) \ < \ N(s_2)(\varnothing, F^>) \ < \ \ldots$$

There also exists an increasing pair $(G^>, H^>)$, with $G^> \neq \varnothing$:

$$N(s_0)(G^>, H^>) \ < \ N(s_1)(G^>, H^>) \ < \ N(s_2)(G^>, H^>) \ < \ \ldots$$

Indeed, if this were not the case, then, by Dickson's Lemma, there would be two nodes $s_i, s_j \in \pi'$ whose profiles satisfy the strict partial order $\lhd$, that is, $N(s_i) \unlhd N(s_j)$ and $N(s_i) \neq N(s_j)$, and this would contradict the assumption that $\mathcal{T}$ was pointwise $\unlhd$-maximal.

We claim that for all increasing pairs $(G, H)$, with $G \neq \varnothing$, and all nodes $s_i \in \pi'$, we have $N(s_i)|_1(G) < \infty$. Indeed, if this were not the case for some increasing pair $(G, H)$ and some node $s_i$, then, since $\mathcal{T}$ is pointwise fair, we would have $N(s_i)(G, H) = \infty$ and hence $(G, H)$ could not be an increasing pair. Now, consider two consecutive nodes $s_i$ and $s_{i+1}$ in $\pi'$ and recall that the slice associated with node $s_i$ has as a sub-slice the slice associated with node $s_{i+1}$. Let $s_i = (\vec{y}_i, \overleftarrow{y}_i)$ and $s_{i+1} = (\vec{y}_{i+1}, \overleftarrow{y}_{i+1})$ be the coordinates of these slices in the underlying decomposition tree. From the existence of the increasing pair $(G^>, H^>)$ and the fact that $N(s_i)|_1(G^>) < \infty$, it follows that the lower rows of the two slices $s_i$ and $s_{i+1}$ cannot coincide, namely, we have $\vec{y}_i < \vec{y}_{i+1}$. Symmetrically, from the existence of the increasing pair $(\varnothing, F^>)$, we deduce that the upper rows of the two slices $s_i$ and $s_{i+1}$ cannot coincide, namely, $\overleftarrow{y}_{i+1} < \overleftarrow{y}_i$. Indeed, by contraposition, if $\overleftarrow{y}_{i+1} = \overleftarrow{y}_i$, then $\vec{y}_i < \vec{y}_{i+1}$ and hence $N(s_{i+1})(\varnothing, F^>) \leq N(s_i)(\varnothing, F^>)$ (a contradition). All together, for all nodes $s_i = (\vec{y}_i, \overleftarrow{y}_i)$ and $s_{i+1} = (\vec{y}_{i+1}, \overleftarrow{y}_{i+1})$ in $\pi'$, we have

$$\vec{y}_i \ < \ \vec{y}_{i+1} \ < \ \overleftarrow{y}_{i+1} \ < \ \overleftarrow{y}_i.$$

The next step consists of generating an infinite sequence of pointwise fair profile trees $\mathcal{T}_0, \mathcal{T}_1, \ldots$ as follows. The first tree $\mathcal{T}_0$ in the sequence is simply the subtree of $\mathcal{T}$ rooted at $s_0$. Each subsequent tree $\mathcal{T}_i$, for $i = 1, 2, \ldots$, is obtained by repeatedly applying to $\mathcal{T}_0$ a suitable *deletion* operation, until the node $s_i$ (identified by its coordinates) becomes a child of the root $s_0$ (in particular, if $s_1$ is already a child of $s_0$ in $\mathcal{T}_0$, then we simply let $\mathcal{T}_1 = \mathcal{T}_0$). Intuitively, the deletion operation receives as input an infinite profile tree $\mathcal{T}'$, with root $r$, and a child $s$ of $r$ that has to be removed, and it returns a new profile tree $\mathcal{T}''$ with the same root as $\mathcal{T}'$ and the siblings and the children of $s$ as children. Formally, if $\mathcal{T}' = (T', N', E')$, then $\mathcal{T}'' = (T'', N'', E'')$, where:

- $T'' = T' \setminus \{s\}$, where the node-to-child relationships are naturally inferred from those of $T'$;
- $N''(s') = N'(s')$ for all $s' \in T''$ (note that the deletion operation does not modify the profile of the root $r$);

– $E''(s') = E'(s')$ for all $s' \in T'' \setminus \{r\}$;
– $E''(r)$ is defined as follows. Suppose that $n$ is the number of children of $r$ in $\mathcal{T}'$, $s$ is the $j$-th children of $r$, and $m$ is the number of children of $s$ in $\mathcal{T}'$. The multiset $E''(r)$ is defined as a "bijective insertion" of $E'(s)$ into $E'(r)$. More precisely, we recall from Definition 5 that an insertion of $E'(s)$ into $E'(r)$ at position $j$ is specified by a multiset function $f : E'(r) \to E'(s)$ that is surjective and satisfies suitable matching conditions between source and target components. Here we can choose such a function $f$ to be also injective, hence a bijection; in this case the outcome of the insertion, as described in Lemma 4, is a multiset $E''(r)$ of $(m+n)$-tuples such that

$$E'(r) = E''(r)|_{\{1,\dots,j\}\cup\{j+m,\dots,n+m\}}$$
$$E'(s) = E''(r)|_{\{j,\dots,j+m\}}$$

(note that the second equality holds in virtue of the fact that $f$ is a bijection). Moreover, by properly choosing the insertion function $f$ above, one can guarantee that the fairness properties are preserved and, in particular, that the resulting tree $\mathcal{T}''$ is pointwise fair whenever the input tree $\mathcal{T}'$ was. Finally, to define the profile tree $\mathcal{T}_i$, for every $i \geq 1$, we start from $\mathcal{T}$ and we repeatedly delete the successor of the root that lies along the access path of $s_i$, until the node $s_i$ (identified by means of its coordinates) is promoted as a child of the root.

Now, observe that in the construction of the pointwise fair profile tree $\mathcal{T}_i$, all nodes $s_j$, with $0 \leq j < i$, have been deleted and their children promoted as children of $s_0$. However, since each of the deleted nodes had other nodes adjacent to it in the original tree, its $y$-coordinates $\vec{y}_j$ and $\bar{y}_j$ are still present, possibly unpaired, in $\mathcal{T}_i$. This allows us to identify, for every $0 \leq j \leq i$, the unique node $\vec{s}_{i,j}$ that is a child of $s_0$ in $\mathcal{T}_i$ and whose lower coordinate coincides with $\vec{y}_j$, i.e., $\vec{s}_{i,j} = (\vec{y}_j, y)$ for some other coordinate $y$. Symmetrically, we can identify the unique node $\bar{s}_{i,j}$ that is a child of $s_0$ in $\mathcal{T}_i$ and whose upper coordinate is $\bar{y}_j$, i.e., $\bar{s}_{i,j} = (y, \bar{y}_j)$ for some coordinate $y$. In addition, we denote by $l_{i,j}$ (resp., $r_{i,j}$) the unique index such that $\vec{s}_{i,j}$ (resp., $\bar{s}_{i,j}$) is the $l_{i,j}$-th (resp., $r_{i,j}$-th) child of $s_0$. Let $\mathcal{T}_i = (T_i, N_i, E_i)$. It is not difficult to see that, for all $i \geq 0$ and all $h \leq k \leq i$,

$$E_i|_{l_{i,h},r_{i,k}} = E_{i+1}|_{l_{i+1,h},r_{i+1,k}} \quad \text{and} \quad E_i|_{r_{i,k},r_{i,h}} = E_{i+1}|_{r_{i+1,k},r_{i+1,h}}.$$

Hence, for every $i \geq k$, it holds that:

$$N_{\vec{y}_h,\bar{y}_k} = E_i|_{l_{i,h},r_{i,k}} \quad \text{and} \quad N_{\bar{y}_k,\bar{y}_h} = E_i|_{r_{i,k},r_{i,h}}.$$

By Lemma 6, for all indices $h \leq k$, both profiles $N_{\vec{y}_h,\bar{y}_k}$ and $N_{\bar{y}_k,\bar{y}_h}$ are feasible. Moreover for our purpose we point out that a very similar property also holds for triples. For all $i \geq 0$ and $h \leq h' \leq k \leq i$ we have

$$E_i|_{l_{i,h},r_{i,h'},r_{i,k}} = E_{i+1}|_{l_{i+1,h},r_{i+1,h'},r_{i+1,k}},$$

thus for every $i \geq k$, it holds that:

$$E_{\vec{y}_h,\bar{y}_{h'},\bar{y}_k} = E_i|_{l_{i,h},r_{i,h'},r_{i,k}}.$$

We now prove that there is a sequence of indices $\mathbb{H} = h_0 < h_1 < \dots$ such that:

**(S1)** $N_{\vec{y}_{h_0},\overleftarrow{y}_{h_1}} \subseteq N_{\vec{y}_{h_0},\overleftarrow{y}_{h_2}} \subseteq \dots$ and each pair $(F,G)$ is either constant, that is, $N_{\vec{y}_{h_0},\overleftarrow{y}_{h_i}}(F,G) = N_{\vec{y}_{h_0},\overleftarrow{y}_{h_{i+1}}}(F,G)$ for all $i \geq 1$, or increasing, that is, $N_{\vec{y}_{h_0},\overleftarrow{y}_{h_i}}(F,G) < N_{\vec{y}_{h_0},\overleftarrow{y}_{h_{i+1}}}(F,G)$ for all $i \geq 1$;

**(S2)** $N_{\vec{y}_{h_1},\overleftarrow{y}_{h_0}} \subseteq N_{\vec{y}_{h_3},\overleftarrow{y}_{h_2}} \subseteq \dots$ and each pair $(F,G)$ is either constant, that is, $N_{\vec{y}_{h_{i+1}},\overleftarrow{y}_{h_i}}(F,G) = N_{\vec{y}_{h_{i+3}},\overleftarrow{y}_{h_{i+2}}}(F,G)$ for all $i \geq 0$, or increasing, that is, $N_{\vec{y}_{h_{i+1}},\overleftarrow{y}_{h_i}}(F,G) < N_{\vec{y}_{h_{i+3}},\overleftarrow{y}_{h_{i+2}}}(F,G)$ for all $i \geq 0$.

A graphical account of $\mathbb{H}$ is given in Figure 7 (sequence (S1) to the left and sequence (S2) to the right). We will later show that such a sequence $\mathbb{H}$ will allow us to build a feasible profile $N$ such that there exists $k$ for which $N(s_k) \lhd N$, thus violating pointwise $\unlhd$-maximality of $\mathcal{T}$.

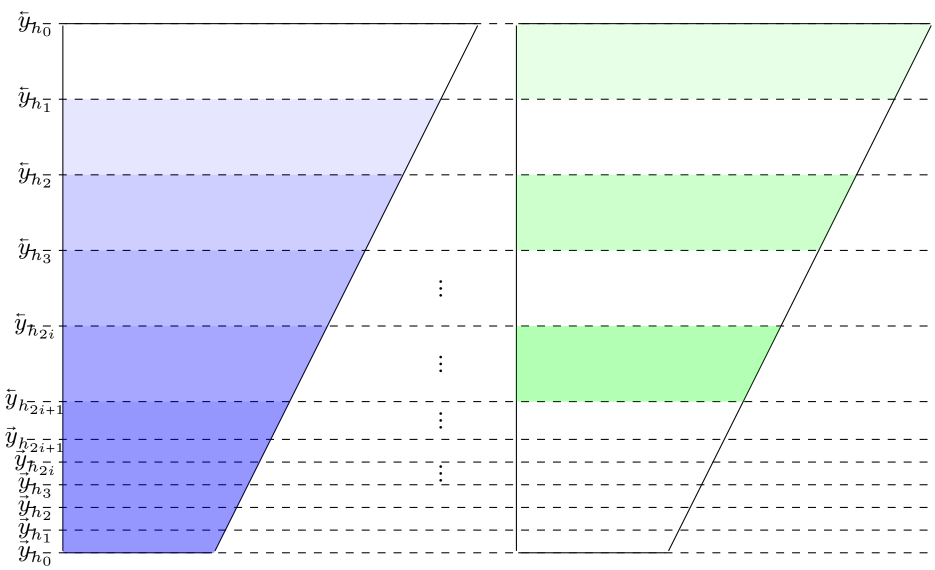

**Fig. 7.** A graphical account of the sequence $\mathbb{H}$

Let $f : \mathbb{N} \to \mathbb{N}$ and $f' : \mathbb{N} \to \mathbb{N}$ be a pair of monotonic, strictly increasing functions over $\mathbb{N}$. We say that $f'$ is a *refinement* of $f$ if $Img(f') \subseteq Img(f)$ and, for every $n \in \mathbb{N}$, $f'(n) \geq f(n)$. We now show how to obtain the sequence $\mathbb{H}$ by an iterative refinement procedure. We start with the identity function $f_0(n) = n$. Now, let $f_i$ be the monotonic strictly increasing function obtained by the $i$-th refinement. If there exist $f_i(0) < f_i(i_1) < f_i(i_2) < \dots$ that satisfy both (S1) and (S2), then we let $h_0 = f_i(0), h_1 = f_i(i_1), h_2 = f_i(i_2), \dots$. Otherwise, we take a subsequence $f_i(0) < f_i(i'_1) < f_i(i'_2) < \dots$ that satisfies (S1) (one such sequence always exists for Dickson's Lemma[4]) and we put $f_{i+1}(n) = f_i(i'_{n+1})$. Termination can be proved as follows. By way of contradiction, suppose that the procedure does not terminates. This means that, for every $i \geq 0$, there is not an infinite subsequence $f_i(0) < f_i(i_1) < f_i(i_2) < \ldots$ in $Img(f_i)$ that satisfy both (S1) and (S2), and thus the procedure selects a subsequence $f_i(0) < f_i(i'_1) < f_i(i'_2) < \ldots$ that satisfies (S1) only, and it uses such a sequence to refine $f_i$ into $f_{i+1}$. Let us consider now the sequence of profiles $N_{\bar{y}_{f_1(0)},\bar{y}_{f_0(0)}}, N_{\bar{y}_{f_3(0)},\bar{y}_{f_2(0)}}, N_{\bar{y}_{f_5(0)},\bar{y}_{f_4(0)}}, \ldots$. By construction, $f_0(0) < f_1(0) < f_2(0) < f_3(0) < f_4(0) < f_5(0) < \ldots$. Then, by Dickson's Lemma, there is a subsequence $j_0 < j_1 < j_2 < \ldots$ of natural numbers, with $f_{j_k}(1) < f_{j_{k+1}}(0)$ for all $k \in \mathbb{N}$, such that $N_{\bar{y}_{f_{j_1}(0)},\bar{y}_{f_{j_0}(0)}} \sqsubseteq N_{\bar{y}_{f_{j_3}(0)},\bar{y}_{f_{j_2}(0)}} \sqsubseteq N_{\bar{y}_{f_{j_5}(0)},\bar{y}_{f_{j_4}(0)}} \sqsubseteq \ldots$ and for each pair $(F, G)$ either $N_{\bar{y}_{f_{j_1}(0)},\bar{y}_{f_{j_0}(0)}}(F, G) = N_{\bar{y}_{f_{j_2}(0)},\bar{y}_{f_{j_3}(0)}}(F, G) = N_{\bar{y}_{f_{j_5}(0)},\bar{y}_{f_{j_4}(0)}}(F, G) = \ldots$ or $N_{\bar{y}_{f_{j_1}(0)},\bar{y}_{f_{j_0}(0)}}(F, G) < N_{\bar{y}_{f_{j_3}(0)},\bar{y}_{f_{j_2}(0)}}(F, G) < N_{\bar{y}_{f_{j_5}(0)},\bar{y}_{f_{j_4}(0)}}(F, G) < \ldots$ Condition (S2) can be satisfied by taking the subsequence $f_{j_0}(0) < f_{j_2}(0) < f_{j_4}(0) < \ldots$ of the sequence $f_{j_0}(0) < f_{j_1}(0) < f_{j_2}(0) < f_{j_3}(0) < f_{j_4}(0) < f_{j_5}(0) < \ldots$. Moreover, by construction, it holds that $Img(f_{j_0}) \supseteq Img(f_{j_1}) \supseteq Img(f_{j_2}) \supseteq \ldots$, and then the sequence $N_{\bar{y}_{f_{j_0}(0)},\bar{y}_{f_{j_1}(0)}}, N_{\bar{y}_{f_{j_0}(0)},\bar{y}_{f_{j_2}(0)}}, N_{\bar{y}_{f_{j_0}(0)},\bar{y}_{f_{j_3}(0)}}, N_{\bar{y}_{f_{j_0}(0)},\bar{y}_{f_{j_4}(0)}}, N_{\bar{y}_{f_{j_0}(0)},\bar{y}_{f_{j_5}(0)}}, \ldots$ satisfies condition (S1). It immediately follows that at the $j_0$-th step there was a subsequence that satisfied both (S1) and (S2), thus forcing the termination of the procedure (contradiction).

Let us now prove that (S1) and (S2) satisfy the following property (*Increase-Anyway*): for the pair $(\varnothing, F^{>})$, either $N_{\bar{y}_{h_1},\bar{y}_{h_0}}(\varnothing, F^{>}) < N_{\bar{y}_{h_3},\bar{y}_{h_2}}(\varnothing, F^{>}) < \ldots$ or there exists an atom $F^{>>}$ such that $N_{\vec{y}_{h_0},\bar{y}_{h_1}}(\varnothing, F^{>>}) < N_{\vec{y}_{h_0},\bar{y}_{h_2}}(\varnothing, F^{>>}) < \ldots$ and $(F^{>>}, F^{>}) \in N_{\bar{y}_{h_1},\bar{y}_{h_0}}$.

Since $N(s_0)(\varnothing, F^{>}) < N(s_1)(\varnothing, F^{>}) < N(s_2)(\varnothing, F^{>}) < \ldots$ and $[(\vec{y}_{h_i}, \bar{y}_{h_i})]_{i \in \mathbb{N}}$ is a subsequence of $[s_i]_{i \in \mathbb{N}}$, it holds that $N(\vec{y}_{h_0}, \bar{y}_{h_0})(\varnothing, F^{>}) < N(\vec{y}_{h_1}, \bar{y}_{h_1})(\varnothing, F^{>}) < N(\vec{y}_{h_2}, \bar{y}_{h_2})(\varnothing, F^{>}) < \ldots$ and thus $N_{\vec{y}_{h_0},\bar{y}_{h_i}}(\varnothing, F^{>}) \geq N_{\bar{y}_{h_i},\bar{y}_{h_i}}(\varnothing, F^{>})$ for every $i \in \mathbb{N}$. Then, we have the following chain $N_{\vec{y}_{h_0},\bar{y}_{h_0}}(\varnothing, F^{>}) < N_{\vec{y}_{h_0},\bar{y}_{h_1}}(\varnothing, F^{>}) < N_{\vec{y}_{h_0},\bar{y}_{h_2}}(\varnothing, F^{>})$.

Let us consider now the sub-sequence $[(\vec{y}_{h_{i+1}}, \bar{y}_{h_i})]_{i \in \mathbb{N} \wedge i \bmod 2 = 0}$, since by construction indexes are drawn from the sequence $h_0 < h_1 < \ldots$ we have $N_{\vec{y}_{h_0},\bar{y}_{h_1}}(\varnothing, F^{>}) < N_{\vec{y}_{h_0},\bar{y}_{h_2}}(\varnothing, F^{>}) < N_{\vec{y}_{h_0},\bar{y}_{h_3}}(\varnothing, F^{>})$ and by condition (S2) $N_{\vec{y}_{h_{i+3}},\bar{y}_{h_{i+2}}}(\varnothing, F^{>}) \geq N_{\vec{y}_{h_{i+1}},\bar{y}_{h_i}}(\varnothing, F^{>})$ for each $i$ then we have the following cases:

1. if for every $i$ it holds that $N_{\vec{y}_{h_{i+1}},\bar{y}_{h_i}}(\varnothing, F^{>}) < N_{\vec{y}_{k_{i+3}},\bar{y}_{h_{i+2}}}(\varnothing, F^{>})$, recall that the property (S2) forces the counters to be either constant or strictly increasing along the sequence. Then property (Increase Anyway) holds;

[4] As a matter of fact, a symmetric argument can be given starting from (S2) instead of (S1).

2. for every $i$ it holds that $N_{\vec{y}_{h_{i+1}},\overleftarrow{y}_{h_i}}(\varnothing, F^>) = N_{\vec{y}_{h_{i+3}},\overleftarrow{y}_{h_{i+2}}}(\varnothing, F^>)$, on the other side we have that $N_{\vec{y}_{h_0},\overleftarrow{y}_{h_i}}(\varnothing, F^>) > N_{\vec{y}_{h_0},\overleftarrow{y}_{h_{i+1}}}(\varnothing, F^>)$ for every $i$ then there exists $F^{>>}$ for which $E_{\vec{y}_{h_0},\vec{y}_{h_{i+3}},\overleftarrow{y}_{h_{i+2}}}(\varnothing, F^{>>}, F^>) > E_{\vec{y}_{h_0},\vec{y}_{h_{i+1}},\overleftarrow{y}_{h_i}}(\varnothing, F^{>>}, F^>)$. Notice that it is only the case because if the triple would have been $(\varnothing, \varnothing, F^>)$ we were in the first case. Then property (Increase Anyway) holds;

On the basis of the existence of the sequence $h_0 < h_1 < \ldots$ satisfying (S1) and (S2), we construct some profiles by taking the limits of the profiles associated with these sequence. More precisely, we define:

$$\vec{N} = \sup_{i\in\mathbb{N}\wedge i>0} N_{\vec{y}_{h_0},\overleftarrow{y}_{h_i}}$$

$$\overleftarrow{N} = \sup_{i\in\mathbb{N}\wedge i \text{ is even}} N_{\vec{y}_{h_{i+1}},\overleftarrow{y}_{h_i}}.$$

Lemma 3 implies that the above profiles are feasible. We can thus let $\vec{\mathcal{T}} = (\vec{\mathcal{T}}, \vec{N}, \vec{E})$ and $\overleftarrow{\mathcal{T}} = (\overleftarrow{\mathcal{T}}, \overleftarrow{N}, \overleftarrow{E})$ be some infinite profile trees that witness the feasibility of $\vec{N}$ and $\overleftarrow{N}$. We prove the following properties:

**(P1)** $\vec{N}|_2 = \overleftarrow{N}|_1$;

**(P2)** for each pair $(F, G)$, with $F \neq \varnothing$, we have $\vec{N}(F, G) = N_{\vec{y}_{h_0},\overleftarrow{y}_{h_1}}(F, G)$;

**(P3)** either $\overleftarrow{N}(\varnothing, F^>) = \infty$ or there exists $F^{>>}$ for which $\vec{N}(\varnothing, F^{>>}) = \infty$ and $(F^{>>}, F^>) \in \overleftarrow{N}$.

The property $(P1)$ trivially holds since both the supremum operations are applied to the two infinite sequence of coordinates $[\overleftarrow{y}_{h_i}]_{i\in\mathbb{N}\wedge i>0}$ and $[\vec{y}_{h_{i+1}}]_{i\in\mathbb{N}\wedge i \text{ is even}}$, where the latter is a sub-sequence of the former. By way of contradiction, suppose that (P2) does not hold, that is, for some pair $(F, G)$ with $F \neq \varnothing$ we have $\vec{N}(F, G) \neq N_{\vec{y}_{h_0},\overleftarrow{y}_{h_1}}(F, G)$. We observe that $\vec{N}(F, G) > N_{\vec{y}_{h_0},\overleftarrow{y}_{h_1}}(F, G)$ since by construction $N_{\vec{y}_{h_0},\overleftarrow{y}_{h_1}}(F, G) \sqsubseteq \vec{N}(F, G)$. Moreover, since the coordinate $\vec{y}_{h_0}$ for the profiles $N_{\vec{y}_{h_0},\overleftarrow{y}_{h_i}}$ is fixed when defining the supremum $\vec{N}$, we have $N_{\vec{y}_h,\overleftarrow{y}_{h+1}}(F, G)|_1 = \vec{N}(F, G)|_1$ and the following two cases may arise:

1. $N_{\vec{y}_{h_0},\overleftarrow{y}_{h+1}}|_1(F) = \infty$ holds. By the fairness property, we have $N_{\vec{y}_{h_0},\overleftarrow{y}_{h+1}}(F, G) = \infty$ and, since $N_{\vec{y}_h,\overleftarrow{y}_{h+1}}(F, G) \sqsubseteq \vec{N}$, we have $N_{\vec{y}_{h_0},\overleftarrow{y}_{h+1}}(F, G) = \vec{N}(F, G) = \infty$ (contradiction).
2. $N_{\vec{y}_{h_0},\overleftarrow{y}_{h+1}}|_1(F) < \infty$. Since the coordinate $\vec{y}_{h_0}$ for the profiles $N_{\vec{y}_{h_0},\overleftarrow{y}_{h_i}}$ is fixed in the definition of supremum $\vec{N}$, we have $N_{\vec{y}_{h_0},\overleftarrow{y}_{h_i}}|_1(F) = \vec{N}|_1(F)$ and hence $\vec{N}(F, G) < \infty$. However, since $N_{\vec{y}_{h_0},\overleftarrow{y}_{h_1}}(F, G) < \vec{N}(F, G)$ holds, (S2) implies $N_{\vec{y}_{h_0},\overleftarrow{y}_{h_1}}(F, G) < N_{\vec{y}_{h_0},\overleftarrow{y}_{h_2}}(F, G) < N_{\vec{y}_{h_0},\overleftarrow{y}_{h_3}}(F, G) < \ldots < \vec{N}(F, G)$, which in its turn implies $\vec{N}(F, G) = \infty$ (contradiction).

Property (P3) trivially holds as a direct consequence of the property (Increase-Anyway).

Let $\mathcal{T}_{h_0} = (T_{h_0}, N_{h_0}, E_{h_0})$ be the tree, according to our previous definition, that corresponds to $h_0$ (i.e., $s_{h_0}$ is a child of $s_0$). We define the multi-set of triples $ET = E_{h_0}|_{l_{i,h_0}, r_{i,h_1}, r_{i,h_0}}$ for some sufficiently large natural number $i$. Clearly, $ET|_{1,3} = N_{\vec{y}_{h_0}, \overleftarrow{y}_{h_0}} = N(s_{h_0})$, $ET|_{1,2} = N_{\vec{y}_{h_0}, \vec{y}_{h_1}}$, and $ET|_{2,3} = N_{\overleftarrow{y}_{h_1}, \overleftarrow{y}_{h_0}}$. Moreover, for each $(F, G)$, with $F \neq \emptyset$, we have, thanks to property (P2), $ET|_{1,2}(F,G) = N_{\vec{y}_{h_0}, \overleftarrow{y}_{h_1}}(F,G) = \vec{N}(F,G)$.

In order to construct a function $E$ inducing a feasible profile that will eventually lead to a contradiction, we define the following multi-sets. Let $Fin$ be the set of triples $(F, G', G) \in ET$ (possibly, $F = \emptyset$) such that $\vec{N}(F, G') < \infty$ and $\overleftarrow{N}(G', G) < \infty$. By the construction of $(S_1)$ and $(S_2)$, $\vec{N}(F, G') < \infty$ implies $ET_{1,2}(F, G') = \vec{N}(F, G')$ and $\overleftarrow{N}(G', G) < \infty$ implies $ET_{2,3}(G', G) = \overleftarrow{N}(G', G)$.

Let $Fin = \{(F_1, G'_1, G_1), \dots (F_n, G'_n, G_n)\}$. For each $1 \le i \le n$, we select a multi-set $EFIN(F_i, G'_i, G_i) \sqsubseteq \overleftarrow{E}$ of $(\overleftarrow{n} + 1)$-tuples. For each $(F_1, \dots, F_{\overleftarrow{n}+1}) \in EFIN(F_i, G'_i, G_i)$, we have $F_1 = G'_i$, $F_{\overleftarrow{n}+1} = G$, and $EFIN(F_i, G'_i, G_i)|_{1,\overleftarrow{n}+1} = ET(F_i, G'_i, G_i)|_{2,3}$.

Let $\vec{s}, \dots, \vec{s}_{\vec{n}}$ be the children of $\vec{r}$ in $\vec{\mathcal{T}}$ and $\overleftarrow{s}_1, \dots \overleftarrow{s}_{\overleftarrow{n}}$ be the children of $\overleftarrow{r}$ in $\overleftarrow{\mathcal{T}}$. We define the following multi-sets of $(\vec{n} + \overleftarrow{n} + 1)$-tuples:

1. For each $(F, G', G) \in ET$, with $F \neq \emptyset$, we distinguish the following cases.
   (a) $ET|_1(F) = \infty$. By fairness, we have $ET|_{1,2}(F, G') = \infty$ and $ET|_{2,3}(G', G) = \infty$. Let $E^{(F,G',G)}$ be a multiset of $(\vec{n}+\overleftarrow{n}+1)$-tuples such that $E^{(F,G',G)} = \{(F_1, \dots, F_{\vec{n}+1}, F_2, \dots, F_{\vec{n}+\overleftarrow{n}+1}) : (F_1, \dots, F_{\vec{n}+1}) \in \vec{E},\ (F_{\vec{n}+1},\ F_2, \dots, F_{\vec{n}+\overleftarrow{n}+1}) \in \overleftarrow{E}, F_1 = F, F_{\vec{n}+1} = G', F_{\vec{n}+\overleftarrow{n}+1} = G\}$. It is easy to show that there exists at least one tuple $(F_1, \dots, F_{\vec{n}+\overleftarrow{n}+1}) \in E^{(F,G',G)}$ for which $\mathsf{req}_{\bar{B}}(F_1) = \bigcup_{2 \le i \le \vec{n}+\overleftarrow{n}+1} \mathsf{obs}(F_i) \cup \mathsf{req}_{\bar{B}}(F_{\vec{n}+\overleftarrow{n}+1})$ and $\mathsf{req}_B(F_{\vec{n}+\overleftarrow{n}+1}) = \bigcup_{1 \le i \le \vec{n}+\overleftarrow{n}} \mathsf{obs}(F_i) \cup \mathsf{req}_B(F_1)$. This guarantees the condition **(profile-infinite-req)** for the tuples $(F_1, F_{\overleftarrow{n}+\vec{n}+1})$ in $E^{(F,G',G)}|_{1,\vec{n}+\overleftarrow{n}+1}$. This condition is sufficient, since, by construction, $E^{(F,G',G)}|_{1,\vec{n}+\overleftarrow{n}+1}(F, G) = \infty$.
   (b) $ET|_{1,2}(F, G') < \infty$ and $\overleftarrow{N}(G', G) < \infty$. Then, by construction, $\vec{N}|_{1,2}(F, G') < \infty$ holds. Let $E^{(F,G',G)}$ be a multiset of $(\vec{n} + \overleftarrow{n} + 1)$-tuples such that $E^{(F,G',G)}|_{1,\dots,\vec{n}+1} = \{(F_1, \dots, F_{\vec{n}+1}) : (F_1, \dots, F_{\vec{n}+1}) \in \vec{E}(\vec{r}), F_1 = F, F_{\vec{n}+1} = G'\}$ and $E^{(F,G',G)}|_{\vec{n}+1,\dots,\vec{n}+1} = EFIN(F, G, G')$. Such a multi-set exists since the cardinalities match. By condition **(profile-finite-req)**, all the tuples $(F_1, \dots, F_{\vec{n}+1}) \in \vec{E}$, with $F_1 = F$ and $F_{\vec{n}+1} = G'$, satisfy $\mathsf{req}_{\bar{B}}(F_1) = \bigcup_{2 \le i \le \vec{n}+1} \mathsf{obs}(F_i) \cup \mathsf{req}_{\bar{B}}(F_{\vec{n}+1})$ and $\mathsf{req}_B(F_{\vec{n}+1}) = \bigcup_{1 \le i \le \vec{n}} \mathsf{obs}(F_i) \cup \mathsf{req}_B(F_1)$. The same holds for all tuples $(F_1, \dots, F_{\overleftarrow{n}+1}) \in \vec{E}$, with $F_1 = F$ and $F_{\overleftarrow{n}+1} = G'$, which thus satisfy $\mathsf{req}_{\bar{B}}(F_1) = \bigcup_{2 \le i \le \overleftarrow{n}+1} \mathsf{obs}(F_i) \cup \mathsf{req}_{\bar{B}}(F_{\overleftarrow{n}+1})$ and $\mathsf{req}_B(F_{\vec{n}+1}) = \bigcup_{1 \le i \le \overleftarrow{n}} \mathsf{obs}(F_i) \cup \mathsf{req}_B(F_1)$. This guarantees that the condition **(profile-finite-req)** of profile trees holds for tuples $(F_1, F_{\vec{n}+\overleftarrow{n}+1})$ in $E^{(F,G',G)}|_{1,\vec{n}+\overleftarrow{n}+1}$;
   (c) $ET|_{1,2}(F, G') < \infty$ and $\overleftarrow{N}|_1(G') = \infty$. By condition **(profile-finite-req)** (case $\overleftarrow{N}(G', G) < \infty$) or by condition **(profile-infinite-req)** (case $\overleftarrow{N}(G', G) = \infty$), there exists a tuple $(F^*_1, \dots, F^*_{\overleftarrow{n}+1}) \in \overleftarrow{E}$ such that $F^*_1 = G'$, $F^*_{\overleftarrow{n}+1} =$

$G$, $\mathsf{req}_{\bar{B}}(F_1^*) = \bigcup_{2 \le i \le \bar{n}+1} \mathsf{obs}(F_i^*) \cup \mathsf{req}_{\bar{B}}(F_{\bar{n}+1})$ and $\mathsf{req}_B(F_{\bar{n}+1}^*) = \bigcup_{1 \le i \le \bar{n}} \mathsf{obs}(F_i^*) \cup \mathsf{req}_B(F_1^*)$, with $F_1^* = G'$ and $F_{\bar{n}+1}^* = G$. Let $E^{(F,G',G)}$ be a multiset of $(\vec{n}+\bar{n}+1)$-tuples such that $E^{(F,G',G)} = \{(F_1', \ldots, F_{\vec{n}+1}', F_2^*, \ldots, F_{\bar{n}+1}^*) : (F_1', \ldots, F_{\vec{n}+1}') \in \vec{E}(\vec{r})\}$. By condition **(profile-finite-req)**, all the tuples $(F_1, \ldots, F_{\bar{n}+1}) \in \bar{E}$, with $F_1 = F$ and $F_{\bar{n}+1} = G'$, satisfy $\mathsf{req}_{\bar{B}}(F_1) = \bigcup_{2 \le i \le \bar{n}+1} \mathsf{obs}(F_i) \cup \mathsf{req}_{\bar{B}}(F_{\bar{n}+1})$ and $\mathsf{req}_B(F_{\bar{n}+1}) = \bigcup_{1 \le i \le \bar{n}} \mathsf{obs}(F_i) \cup \mathsf{req}_B(F_1)$. The fact that the condition **(profile-finite-req)** holds for the tuples $(F_1, F_{\vec{n}+\bar{n}+1})$ in $E^{(F,G',G)}|_{1,\vec{n}+\bar{n}+1}$ immediately follows.

2. For each $(\varnothing, G', G) \in ET$, we distinguish the following cases.
   (a) $\vec{N}(\varnothing, G') < \infty$ and $\bar{N}(G', G) = \infty$. By case 1.a, all the tuples $(F_1, \ldots, F_{\vec{n}+1})$, with $F_1 = G$ and $F_{\vec{n}+1} = G'$, already appear infinitely often in $E^{(F,G',G)}|_{\vec{n}+1,\ldots,\vec{n}+\bar{n}+1}$. We take one such tuple $(F_1', \ldots, F_{\bar{n}+1}') \in \bar{E}(\bar{r})$, and we define $E^{(\varnothing,G',G)}$ as the multiset with $E^{(\varnothing,G',G)} = \{(\varnothing, F_2, \ldots, F_{\vec{n}+1}, F_2', \ldots, F_{\bar{n}+1}') : (\varnothing, F_2, \ldots, F_{\vec{n}+1}) \in \vec{E}, F_{\vec{n}+1} = G'\}$.
   (b) $\vec{N}(\varnothing, G') < \infty$, $\bar{N}(G', G) < \infty$, and $\vec{N}(\varnothing, G') = ET|_{1,2}$. Let $E^{(\varnothing,G',G)}$ be a multiset of $(\vec{n} + \bar{n} + 1)$-tuples such that $E^{(\varnothing,G',G)}|_{1,\ldots,\vec{n}+1} = \{(\varnothing, \ldots, F_{\vec{n}+1}) : (\varnothing, \ldots, F_{\vec{n}+1}) \in \vec{E}(\vec{r}), F_{\vec{n}+1} = G'\}$ and $E^{(\varnothing,G',G)}|_{\vec{n}+1,\ldots,\vec{n}+\bar{n}+1} = EFIN(\varnothing, G', G)$.
   (c) $\vec{N}(\varnothing, G') = \infty$ and $\bar{N}(G', G) < \infty$. We define $E^{(\varnothing,G',G)}$ as a multiset of $(\vec{n} + \bar{n} + 1)$-tuples such that $E^{(\varnothing,G',G)}|_{1,\ldots,\vec{n}+1} = \{(\varnothing, \ldots, F_{\vec{n}+1}) : (\varnothing, \ldots, F_{\vec{n}+1}) \in \vec{E}(\vec{r}), F_{\vec{n}+1} = G'\}$ and $E^{(\varnothing,G',G)}|_{\vec{n}+1,\ldots,\vec{n}+\bar{n}+1} \sqsupseteq \{(F_1, \ldots, F_{\bar{n}+1}) : (F_1, \ldots, F_{\bar{n}+1}) \in \bar{E}, F_1 = G', F_{\bar{n}+1} = G\}$.
   (d) $\vec{N}(\varnothing, G') = \infty$ and $\bar{N}(G', G) = \infty$. We define $E^{(\varnothing,G',G)}$ as a multiset of $(\vec{n} + \bar{n} + 1)$-tuples such that $E^{(\varnothing,G',G)}|_{1,\ldots,\vec{n}+1} = \{(\varnothing, \ldots, F_{\vec{n}+1}) : (F_1, \ldots, F_{\vec{n}+1}) \in \vec{E}(\vec{r}), F_{\vec{n}+1} = G'\}$ and $E^{(\varnothing,G',G)}|_{\vec{n}+1,\ldots,\vec{n}+\bar{n}+1} = \{(F_1, \ldots, F_{\bar{n}+1}) : (F_1, \ldots, F_{\bar{n}+1}) \in \bar{E}(\bar{r}), F_1 = G', F_{\bar{n}+1} = G\}$.

3. For each tuple $(G', G) \in \bar{N}$, with $\bar{N}(G', G) = \infty$, $ET(F, G', G) < \infty$ for all $F \neq \varnothing$, and $(\varnothing, G', G) \notin ET$, we proceed as follows. If there were $F \neq \varnothing$ such that $ET(F, G') = \infty$, by fairness we would have $ET(F, G', G) = \infty$ (contradiction). Hence, for all $i \ge 0$ and all $F \neq \varnothing$, $N_{\vec{y}_k, \bar{y}_{h_i}}(F, G') = N_{\vec{y}_k, \bar{y}_{h_{i+1}}}(F, G')$, and thus $ET(F, G') = \vec{N}(F, G') < \infty$. Since, by property P1, $\vec{N}|_2(G') = \bar{N}(G')|_1$ and $\bar{N}_1(G') = \infty$, there exists $(\varnothing, G') \in \vec{N}$ such that $\vec{N}(\varnothing, G') = \infty$. Thus we can define the multiset $E_{fix}^{(\varnothing,G',G)}$ of $(\vec{n} + \bar{n} + 1)$-tuples such that $E_{fix}^{(\varnothing,G',G)}|_{1,\ldots,\vec{n}+1} = \{(\varnothing, \ldots, F_{\vec{n}+1}) : (\varnothing, \ldots, F_{\vec{n}+1}) \in \vec{E}_\infty(\vec{r}), F_{\vec{n}+1} = G'\}$ and $E_{fix}^{(\varnothing,G',G)}|_{\vec{n}+1,\ldots,\vec{n}+\bar{n}+1} = \{(F_1, \ldots, F_{\bar{n}+1}) : (F_1, \ldots, F_{\bar{n}+1}) \in \bar{E}(\bar{r}), F_1 = G', F_{\bar{n}+1} = G\}$, where $\vec{E}_\infty(\vec{r})$ is the restriction of the multiset $\vec{E}(\vec{r})$ to the tuples that appear with infinite multiplicity (such multisets can be found by exploiting simple counting arguments, since $\vec{N}(\varnothing, G') = \infty$, and hence $E_{fix}^{(\varnothing,G',G)}|_{1,\ldots,\vec{n}+1} \neq \varnothing$).

4. For each tuple $(\emptyset, G) \in \overleftarrow{N}$, we define the multiset of $(\vec{n}+\overleftarrow{n}+1)$-tuples such that $E^{(\emptyset,\emptyset,G)} = \{(\emptyset, \dots, \emptyset, F_2, \dots, F_{\vec{n}+\overleftarrow{n}+1}) : (\emptyset, F_2, \dots, F_{\overleftarrow{n}+1}) \in \overleftarrow{E}(\overleftarrow{r}), F_{\vec{n}+1} = G\}$.

Now, towards a conclusion, let $E$ be the multiset of $(\vec{n}+\overleftarrow{n}+1)$-tuples obtained from the union of all the multisets defined at points 1.,2.,3., and 4. By construction, we have the two profiles $E|_{i,i+1}$ and $\vec{E}_{i,i+1}$ coincide for all $1 \le i \le \overleftarrow{n}$, and thus they are feasible. Moreover, by the definitions in point 4., we have $\overleftarrow{E}|_{i,i+1} \sqsubseteq E_{i,i+1}$ for all $\vec{n}+1 \le i \le \vec{n}+\overleftarrow{n}+1$, and $E|_{i,i+1}(\emptyset, G) = \overleftarrow{E}|_{i,i+1}(\emptyset, G)$ for all $G$. By the above properties and Lemma 1, the profile $E|_{i,i+1}$ is feasible, and hence $E|_{1,\overleftarrow{n}+\vec{n}+1}$ is feasible. By the definitions in point 1., we have that $E|_{1,\overleftarrow{n}+\vec{n}+1}(F, G) = ET|_{1,3}(F, G)$ for all $(F, G) \in ET|_{1,3}$, with $F \neq \emptyset$. Since $ET|_3 \sqsubseteq \overleftarrow{N}_2 \sqsubseteq E|_{\overleftarrow{n}+\vec{n}+1}$, we also know that $ET \trianglelefteq E|_{1,\vec{n}+\overleftarrow{n}+1}$. Moreover, by construction and the property (Increase Anyway), we have that $E|_{\vec{n}+1,\vec{n}+\overleftarrow{n}+1}(\emptyset, F^{>}) = \infty$ or $E|_{1,\vec{n}+1}(\emptyset, F^{>>}) = \infty$. In the former case we can immediately conclude that $E|_{1,\vec{n}+\overleftarrow{n}+1}(\emptyset, F^{>}) = \infty$, in the latter we have that for one of the construction done in point $2(c)$, $2(d)$ or 3 implies $E|_{1,\vec{n}+1,\vec{n}+\overleftarrow{n}+1}(\emptyset, F^{>>}, F^{>}) = \infty$ and thus $E|_{1,\vec{n}+\overleftarrow{n}+1}(\emptyset, F^{>}) = \infty$ as well. Having $E|_{1,\vec{n}+\overleftarrow{n}+1}(\emptyset, F^{>}) = \infty$ allows us to assume $ET|_{1,3}(\emptyset, F^{>}) < E|_{1,\vec{n}+\overleftarrow{n}+1}(\emptyset, F^{>})$. This implies $ET|_{1,3} \triangleleft E|_{1,\overleftarrow{n}+\vec{n}+1}$. Finally, recall that $ET|_{1,3} = N_{\vec{y}_k,\overleftarrow{y}_k} = N(s_k)$, where $s_k$ was a node in the original profile tree. We thus have $N(s_k) \triangleleft E|_{1,\overleftarrow{n}+\vec{n}+1}$, which contradicts the hypothesis that $\mathcal{T}$ was pointwise $\trianglelefteq$-maximal. ☐

**Theorem 2.** *The satisfiability problem for* $\mathsf{A\bar{A}B\bar{B}}$ *interpreted over* $\mathbb{Q}$*, as well as over the class of all linear orders, is decidable, but not primitive recursive.*

*Proof.* In Section 4 we described two semi-decision procedures that, together, solve the satisfiability problems over $\mathbb{Q}$ and over the class of all interval structures. The satisfiability problems are thus shown to be non-primitive recursive.

It now remains to prove the complexity lower-bound, which follows a reduction from a variant of the reachability problem for lossy counter machines (see beginning of proof of Theorem 1 for a definition of lossy counter machine). Specifically, we fix a lossy counter machine $\mathcal{M} = (Q, k, \delta)$ and two control states $q_{\mathsf{init}}, q_{\mathsf{halt}}$ and we construct a formula $\varphi^{\mathcal{M}}$ that describes precisely the *non-termination property* of $\mathcal{M}$, that is, the fact that the machine $\mathcal{M}$ admits an infinite computation that starts in the configuration $(q_{\mathsf{init}}, \bar{z}_0)$, with $\bar{z}_0 = (0, \dots, 0)$, and avoids the halting state $q_{\mathsf{halt}}$. In [13] the (non-)termination problem is shown to have strictly non-primitive recursive complexity.

For the sake of simplicity, we begin by assuming that the underlying temporal domain is isomorphic to $\mathbb{N}$. In this case the formula $\varphi^{\mathcal{M}}$ is very similar to that of Theorem 1:

$$\begin{aligned}
\varphi^{\mathcal{M}} = {} & [\mathsf{G}]\Big(\langle\mathsf{A}\rangle\mathsf{true}\wedge\langle\bar{\mathsf{A}}\rangle\mathsf{true}\wedge[\mathsf{B}]\mathsf{false} \;\rightarrow\; \bigvee_{a\,\in\,Q\cup C} a \;\wedge \bigwedge_{a\neq b\,\in\,Q\cup C}\neg(a\wedge b)\Big) \;\wedge \\
& [\mathsf{G}]\Big([\mathsf{A}]\mathsf{false}\vee[\bar{\mathsf{A}}]\mathsf{false}\vee\langle\mathsf{B}\rangle\mathsf{true} \;\rightarrow \bigwedge_{a\,\in\,Q\cup C\cup\{\mathsf{inc},\mathsf{dec}\}}\neg a\Big) \;\wedge \\
& [\mathsf{G}]\bigwedge_{q\,\in\,Q}\Big(\langle\mathsf{B}\rangle q \;\rightarrow\; \langle\bar{\mathsf{A}}\rangle\langle\mathsf{A}\rangle\varphi^{\delta}_{q}\Big) \;\wedge\; [\mathsf{G}]\neg q_{\mathsf{halt}} \;\wedge \\
& \boldsymbol{\langle\mathsf{G}\rangle\Big(q_{\mathsf{init}} \;\wedge\; [\mathsf{A}]\big(\,\langle\mathsf{A}\rangle \bigvee_{c\in C} c\rightarrow\langle\mathsf{B}\rangle\langle\mathsf{B}\rangle\mathsf{true}\,\big)\Big)}\,.
\end{aligned}$$

where the letters in $Q$ are used to represent the control states of $\mathcal{M}$, the letters in $C = \{c_1, \ldots, c_k\}$ are used to represent the register names of $\mathcal{M}$, the letter $\mathsf{inc}$ (resp., $\mathsf{dec}$) is used to mark a single unit-length subinterval in each block that follows an increment operation (resp., precedes a decrement operation) on the corresponding register, and, finally, the subformula $\varphi^{\delta}_{q}$ is used to enforce the possible transitions of $\mathcal{M}$. The only difference with the definitions of Theorem 1 is highlighted in bold in the last line of the formula: beside enforcing that the run starts in the control state $q_{\mathsf{init}}$, we also require that the values of the counters are all 0 (formally, we require that the first unit-interval to the right of $q_{\mathsf{init}}$ cannot be labelled with the name of a counter).

Clearly, the above formula $\varphi^{\mathcal{M}}$ is satisfied in some interval structure with $\mathbb{N}$ as temporal domain iff there exists an infinite computation of $\mathcal{M}$ of the form $(q_0, \bar{z}_0)\ (q_1, \bar{z}_1)\ldots$, with $q_0 = q_{\mathsf{init}}$, $\bar{z}_0 = (0, \ldots, 0)$, and $q_i \neq q_{\mathsf{halt}}$ for all $i \geq 1$.

To conclude the proof we need to explain how to turn $\varphi^{\mathcal{M}}$ into an analogous formula that defines non-termination of $\mathcal{M}$ over the temporal domain $\mathbb{Q}$. We use again a technique from the proof of Theorem 1, namely, we embed a discrete linear ordering inside $\mathbb{Q}$ by means of a distinguished propositional letter $\#$. However, differently from the previous case, where we could exploit the Dedekind-completeness of the temporal domain, here we will not be able to embed an isomorphic copy of $\mathbb{N}$ inside $\mathbb{Q}$. On the other hand, we observe that the above reduction via $\varphi^{\mathcal{M}}$ is correct even for a temporal domain that is not isomorphic to $\mathbb{N}$, but $\mathbb{Z}$-*like*, in the sense that all elements in it have an immediate predecessor and an immediate successor. Example of $\mathbb{Z}$-like orderings are $\mathbb{Z}$, $\mathbb{Z}^2$, $\mathbb{Z}^{\omega}$, etc. Moreover, embeddings of $\mathbb{Z}$-like orderings inside $\mathbb{Q}$ can be defined by means of the following simple formula:

$$\begin{aligned}
\varphi_{\#} \;=\; & [\mathsf{G}]\big(\# \;\rightarrow\; \pi\big) \;\wedge \\
& [\mathsf{G}]\Big(\# \;\rightarrow\; \langle\mathsf{A}\rangle\big(\langle\mathsf{A}\rangle\#\wedge[\mathsf{B}][\mathsf{A}]\neg\#\big)\Big) \;\wedge \\
& [\mathsf{G}]\Big(\# \;\rightarrow\; \langle\bar{\mathsf{A}}\rangle\big(\langle\bar{\mathsf{A}}\rangle\#\wedge[\mathsf{B}][\mathsf{A}]\neg\#\big)\Big)\,.
\end{aligned}$$

Finally, in order to correctly express non-termination of $\mathcal{M}$ over the rationals it is sufficient to restrict the range of the quantifications of the formula $\varphi^{\mathcal{M}}$ over the intervals satisfying $\langle\mathsf{B}\rangle\#\wedge\langle\mathsf{A}\rangle\#$. $\square$