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# Residually stressed beams

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ABSTRACT

In this paper we derive a theory for a linearly elastic residually stressed rod trough an asymptotic analysis based on Γ-convergence.

## 1 Introduction

The theory of linear elasticity with residual stress goes back to Cauchy (1829), but for a long time the attention of researchers was almost exclusively given to the so-called linear theory of elasticity. In recent years, instead, the theory with residual stress has been studied and used quite extensively, see [2, 6, 16, 17, 19, 20, 22, 23, 24, 25, 28, 29, 30] and references therein.

The aim of the present paper is to deduce, by means of Γ-convergence, a variational model for slender rods with residual stress. Beam theories for a linear elastic material without residual stress have been derived, by Γ-convergence, in [3, 7, 8, 9, 10, 11, 26].

The presence of residual stress introduces in the constitutive equation for the Piola-Kirchhoff stress tensor a dependence from the displacement gradient and not simply on the strain as in the case without residual stress. Precisely, the Piola-Kirchhoff stress tensor $S$ is given by

$$S = \mathring{T} + Du\mathring{T} + \mathbb{L}Eu,$$

where $Du$ denotes the gradient of the displacement $u$, $Eu$ is the symmetric part of $Du$, $\mathring{T}$ is a second order symmetric tensor representing the residual stress in the reference configuration and $\mathbb{L}$ is a fourth order tensor called incremental elasticity tensor. The term $Du\mathring{T}$, that comes into play because of material frame indifference, makes the theory quite different from the elastic theory without residual stress; for instance, the elastic energy density is no longer convex.

In our analysis we do not impose any material symmetry on the incremental elasticity tensor $\mathbb{L}$ and we allow it to depend on the longitudinal variable $y_3$, i.e., the cross-sections of the beams are assumed to be homogeneous. By assuming

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the rod to be clamped to one of its bases, we find that the elastic energy of the limit problem (see Theorem 6.2 and (47)) is

$$I_{1\mathrm{d}}(\xi,\vartheta)=\frac{1}{2}\int_0^\ell Q(y_3,\xi_1'',\xi_2'',\xi_3',\vartheta')+\mathrm{tr}\langle\mathring{T}\rangle\,\vartheta^2+\langle\mathring{T}\rangle(\xi_1',\xi_2')\cdot(\xi_1',\xi_2')\,dy_3,$$

where $\xi$ is a Bernoulli-Navier displacement and $\vartheta$ is a scalar field representing the twist of the cross-section around the longitudinal axis. The energy density $Q$ is defined by a minimum problem on the cross-section, see (45), involving the incremental elasticity tensor $\mathbb{L}$, and

$$\langle\mathring{T}\rangle:=\int_\omega\begin{pmatrix}\mathring{T}_{11} & \mathring{T}_{12}\\ \mathring{T}_{21} & \mathring{T}_{22}\end{pmatrix}dy_1dy_2,$$

where $\omega$ denotes the cross-section.

The paper is organized as follows. In Section 2 we introduce the equilibrium problem for an elastic rod with residual stress and state some properties implied by the equilibrium equations on the residual stress. The reference configuration of the body is assumed to be a cylinder with small ratio between diameter of the cross-section and length. To find a 1d approximation of this problem, in Section 3 we introduce a sequence of three-dimensional problems on cylinders whose diameters are proportional to a parameter approaching zero. The existence of a solution is also discussed. In Section 4, following the idea of Ciarlet and Destuynder [4], we re-scale the sequence of three-dimensional problems to a fixed domain. In Section 5 we study the compactness properties of sequences of displacements with equi-bounded energy and in Section 6 we state and prove the $\Gamma$-convergence result and the convergences of minima and minimizers. The paper ends with a small section in which we discuss the problem defining $Q$.

**Notation.** Repeated Latin indices are summed from 1 to 3 while repeated Greek indices are summed from 1 to 2. The gradient (i.e. the Jacobian matrix) is denoted by $D$ and $D_i$ will denote the derivative with respect to the $i$-th variable. The notation used for Lebesgue and Sobolev spaces is standard (see Adams [1]) while the notation used to describe the operations on tensorial quantities is similar to that used by Gurtin [14]. Convergence in the norm will be denoted by $\to$ while weak convergence is denoted by $\rightharpoonup$.

# 2 Elastic rods with residual stress

In this Section we introduce the equilibrium problem for an elastic rod with residual stress and we study some of the restrictions imposed by the equilibrium equations on the residual stress; this characterization will be of some use in Section 6.

Let

$$\Omega:=\omega\times(0,\ell)$$

where $\omega$ is a connected, simply connected, bounded, open subset of $\mathbb{R}^2$ with Lipschitz boundary. We denote by $S(x_3):=\omega\times\{x_3\}$ for any $x_3\in[0,\ell]$. Hereafter we take $x_1$, $x_2$ central principal axes of inertia.

We shall refer to $\Omega$ as a residually stressed reference configuration of an elastic body, that is: there is a (residual) stress $\mathring{T}$ not identically equal to zero that satisfies the equilibrium equations in the absence of external actions

$$\begin{cases} \operatorname{div} \mathring{T} = 0 & \text{in } \Omega, \\ \mathring{T} = (\mathring{T})^T & \text{in } \Omega, \\ \mathring{T} n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $n$ is the outward unit normal to the boundary of $\Omega$.

In what follows we consider a fully anisotropic material which is also allowed to be inhomogeneous along the $x_3$-axis, so that the first Piola-Kirchhoff stress field $S$ can be expressed, see [17, 19, 24, 25], as

$$S(x) = \mathring{T}(x) + Du(x)\mathring{T}(x) + \mathbb{L}(x_3)Eu(x)$$

where $Du$ denotes the gradient of the displacement $u$,

$$Eu := \text{sym } Du = \frac{Du + (Du)^T}{2},$$

is the strain, and $\mathbb{L}(x_3)$ is the incremental elasticity tensor evaluated at the cross-section of coordinate $x_3$.

We assume $\mathbb{L}$ to be essentially bounded,

$$\mathbb{L} \in L^\infty((0,\ell); \mathbb{R}^{3\times3\times3\times3}),$$

to have the major and minor symmetries,

$$\mathbb{L}_{ijkl} = \mathbb{L}_{jikl} = \mathbb{L}_{klji},$$

and to be positive definite,

$$\exists\, C > 0, \text{ s.t. } \mathbb{L}(x_3)A \cdot A \geq C|A|^2, \tag{2}$$

for all $A \in \mathbb{R}^{3\times3}_{\text{sym}} := \{A \in \mathbb{R}^{3\times3} : A = A^T\}$ and for a.e. $x_3 \in (0,\ell)$. We denote by $C_L$ the largest of all such constants $C$. Furthermore we assume $\mathring{T} \in L^\infty(\Omega; \mathbb{R}^{3\times3}_{\text{sym}})$. From this assumption and from the first equation of (1) we deduce that $\mathring{T}$ and div$\mathring{T}$ are square integrable fields, hence (see Girault and Raviart [13, equation (2,17)]) the normal trace to the boundary of $\mathring{T}$ is well defined and in particular the third equation of (1) makes sense.

We consider the body clamped on $S(0)$, and subjected to dead body forces $\tilde{b} \in L^2(\Omega; \mathbb{R}^3)$. The weak form of the equilibrium problem is: find $u \in H^1_\flat(\Omega; \mathbb{R}^3)$ such that

$$\int_\Omega Du\mathring{T} \cdot Dv + \mathbb{L}Eu \cdot Ev \, dx = \int_\Omega \tilde{b} \cdot v \, dx, \tag{3}$$

for all $v \in H^1_\flat(\Omega; \mathbb{R}^3)$, where

$$H^1_\flat(\Omega; \mathbb{R}^3) := \{u \in H^1(\Omega; \mathbb{R}^3) : u = 0 \text{ on } S(0)\}.$$

We conclude this section by deducing some properties of the residual stress field which will be used in Section 6. These properties follow from (1) and from the particular (cylindrical) geometry of the reference configuration.

**Lemma 2.1.** *Let $\mathring{T} \in L^\infty(\Omega;\mathbb{R}^{3\times3}_{\rm sym})$ be a field that satisfies the equilibrium equations* (1) *in $\Omega = \omega \times (0,\ell)$. Then we have*

1. $\displaystyle\int_\omega \mathring{T}_{i3}\,dx_1dx_2 = 0$ *for a.e.* $x_3 \in (0,\ell)$*;*

2. $\displaystyle\int_\omega \mathring{T}_{\alpha\beta}\,dx_1dx_2 = \frac{\partial}{\partial x_3}\Big(\int_\omega x_\beta \mathring{T}_{\alpha 3}\,dx_1dx_2\Big)$ *for a.e.* $x_3 \in (0,\ell)$,

*for $i = 1,2,3$ and $\alpha,\beta = 1,2$.*

*Proof.* Since $\mathring{T}$ and ${\rm div}\mathring{T}$ are square integrable fields, we are in position to use Green's identity (see Girault and Raviart [13, equation (2,17)]) to deduce, from (1), that

$$\int_\Omega \mathring{T}\cdot D\varphi\,dx = 0 \quad \forall\,\varphi \in H^1(\Omega;\mathbb{R}^3). \tag{4}$$

Using text functions depending on $x_3$ only, that is with the choice $\varphi(x) = \varphi(x_3)$, we have

$$\int_\Omega \mathring{T}_{31}\varphi_1' + \mathring{T}_{32}\varphi_2' + \mathring{T}_{33}\varphi_3'\,dx = 0.$$

Let us now fix $i \in \{1,2,3\}$ and, for every $g \in C_c^\infty(0,\ell)$, we take $\varphi_i(x_3) = \int_0^{x_3} g\,ds$ and set the other two components of $\varphi$ to be identically equal to 0, to deduce that

$$\int_0^\ell \int_\omega \mathring{T}_{3i}\,dx_1dx_2\,g(x_3)dx_3 = 0 \quad \forall\,g \in C_c^\infty(0,\ell).$$

This implies $\int_\omega \mathring{T}_{3i}\,dx_1dx_2 = 0$ for a.e. $x_3 \in (0,\ell)$, and hence claim *1* follows by symmetry.

To prove *2*, having fixed $i \in \{1,2\}$, we take in (4) text functions of the form $\varphi_i = x_\alpha f(x_3)$ with $f \in C_c^\infty(0,\ell)$, $\alpha \in \{1,2\}$ and we set the other two components of $\varphi$ to be identically equal to 0. This leads to

$$\int_\Omega \mathring{T}_{i1}\delta_{\alpha1}f + \mathring{T}_{i2}\delta_{\alpha2}f + \mathring{T}_{i3}x_\alpha f'\,dx = 0 \quad \forall\,f \in C_c^\infty(0,\ell),$$

where $\delta$ denotes the Kronecker's symbol. We therefore have deduced that

$$\int_0^\ell \Big(\int_\omega \big(\mathring{T}_{i1}\delta_{\alpha1}+\mathring{T}_{i2}\delta_{\alpha2}\big)\,dx_1dx_2\,f + \int_\omega \mathring{T}_{i3}x_\alpha\,dx_1dx_2 f'\Big)\,dx_3 = 0\;\forall\,f \in C_c^\infty(0,\ell),$$

and an integration by parts concludes the proof. □

# 3 A sequence of problems

The aim of our investigation is to provide a 1d model that approximates the problem laid down in Section 2 when the ratio [diameter of $\omega$]/$\ell$ is small.

To this aim, we introduce in this section a sequence of three-dimensional problems parametrized by a parameter $\varepsilon \in (0,1]$ such that the element of the sequence corresponding to $\varepsilon = 1$ coincides with the problem of Section 2. The sequence chosen will $\Gamma$-converge as $\varepsilon \to 0$, the asymptotic analysis will be the aim of the subsequent sections, and the $\Gamma$-limit will be the 1d approximate problem.

For all $\varepsilon \in (0,1]$ and $\ell > 0$ let

$$\Omega_\varepsilon := \omega_\varepsilon \times (0, \ell),$$

where $\omega_\varepsilon := \varepsilon\omega$. We denote by $S_\varepsilon(x_3) := \omega_\varepsilon \times \{x_3\}$ for any $x_3 \in [0, \ell]$.

On each domain $\Omega_\varepsilon$ we consider a problem of the same kind of (3), precisely: find $u \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$ such that

$$\int_{\Omega_\varepsilon} Du\mathring{T}^\varepsilon \cdot Dv + \mathbb{L}Eu \cdot Ev \, dx = \int_{\Omega_\varepsilon} b^\varepsilon \cdot v \, dx, \tag{5}$$

for all $v \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$, where

$$H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3) := \{u \in H^1(\Omega_\varepsilon; \mathbb{R}^3) : u = 0 \text{ on } S_\varepsilon(0)\},$$

and where the sequences $\mathring{T}^\varepsilon \in L^\infty(\Omega_\varepsilon; \mathbb{R}^{3\times 3}_{\rm sym})$ and $b^\varepsilon \in L^2(\Omega_\varepsilon; \mathbb{R}^3)$ will be specified in the next section. According to the promise made at the beginning of the section we shall have $\mathring{T}^1 = \mathring{T}$ and $b^1 = \tilde{b}$.

We will not assume the sequence of tensor fields $\mathring{T}^\varepsilon$ to be divergence-free. This will leave us more freedom in the choice of the scaling of this term, which will be done in Section 4. The effect of a different choice, i.e., the introduction of a divergence-free condition on $\mathring{T}^\varepsilon$, will be discussed in Remark 6.5.

Let us now discuss the existence of a solution $u$ of (5) following the lines traced in [22]. To this aim, a crucial role is played by Korn's inequality (see Anzellotti, Baldo and Percivale [3], Theorem A.1, and for a simpler proof see [23]).

**Theorem 3.1.** *There exists a constant $C > 0$, independent of $\varepsilon$, such that*

$$\int_{\Omega_\varepsilon} |u|^2 + |Du|^2 dx \le \frac{C}{\varepsilon^2} \int_{\Omega_\varepsilon} |Eu|^2 dx, \tag{6}$$

*for every $u \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$ and for every $\varepsilon \in (0,1]$.*

We denote by $C_K$ the smallest constant for which the inequality

$$\int_{\Omega_\varepsilon} |Du|^2 dx \le \frac{C_K}{\varepsilon^2} \int_{\Omega_\varepsilon} |Eu|^2 dx \tag{7}$$

holds true for every $u \in H^1_\flat(\Omega_\varepsilon; \mathbb{R}^3)$ and for every $\varepsilon \in (0,1]$.

**Lemma 3.2.** *Let $S \in \mathbb{R}^{3\times 3}_{sym}$ and $\lambda_m$ its smallest eigenvalue. Then, for all $A \in \mathbb{R}^{3\times 3}$ it holds that*

$$AS \cdot A \ge \lambda_m |A|^2.$$

*Proof.* It is sufficient to write down the components of $S$ and $A$ in the orthonormal basis $\{\mathrm{e}_i\}_{i=1}^3$ of $\mathbb{R}^3$ that diagonalizes $S$. Let $\lambda_i$ be the eigenvalues of $S$, and $A_{ij}$ be the components of $A$ in the basis $\{\mathrm{e}_i\}_{i=1}^3$. Then

$$AS \cdot A = \sum_{i,l=1}^{3} A_{li}^2 \lambda_i \ge \lambda_m \sum_{i,l=1}^{3} A_{li}^2 = \lambda_m |A|^2.$$

□

Let

$$\mathring{\tau}_m^\varepsilon := \operatorname*{essinf}_{x\in\Omega_\varepsilon} \min_{A\in\mathbb{R}^3} \{\mathring{T}^\varepsilon(x)A\cdot A \;:\; |A| = 1\}, \tag{8}$$

denote the essential infimum of the smallest eigenvalue of $\mathring{T}^\varepsilon$. Of course, for a generic $\mathring{T}^\varepsilon \in L^\infty(\Omega_\varepsilon;\mathbb{R}^{3\times3}_{\mathrm{sym}})$ the bilinear form in the first member of (5) is not $H^1$-coercive. This lack of coercivity occurs also if $\mathring{T}^\varepsilon$ has the physical meaning of a residual stress tensor, that is if it satisfies (1); indeed, in the latter case, it can be shown that $\mathring{\tau}_m^\varepsilon$ is either identically equal to 0 or that it also takes negative values (see [22]). Therefore, to prove existence and uniqueness of the solution of problem (5), we shall suppose that the absolute value of $\mathring{\tau}_m^\varepsilon$ is small enough, that is: the compressions due to $\mathring{T}^\varepsilon$ are not too large.

**Theorem 3.3.** *Assume that*

$$C_L > C_K\frac{|\mathring{\tau}_m^\varepsilon|}{\varepsilon^2}. \tag{9}$$

*Then there exists a unique solution $u^\varepsilon \in H^1_\flat(\Omega_\varepsilon;\mathbb{R}^3)$ of problem (5).*

*Proof.* From (2), (7), (8) and Lemma 3.2 we have, for any $v \in H^1_\flat(\Omega_\varepsilon;\mathbb{R}^3)$,

$$\begin{aligned}\int_{\Omega_\varepsilon} Dv\mathring{T}^\varepsilon\cdot Dv + \mathbb{L}Ev\cdot Ev\,dx &\geq \mathring{\tau}_m^\varepsilon\|Dv\|^2_{L^2(\Omega_\varepsilon)} + C_L\|Ev\|^2_{L^2(\Omega_\varepsilon)}\\ &\geq \left(C_L - C_K\frac{|\mathring{\tau}_m^\varepsilon|}{\varepsilon^2}\right)\|Ev\|^2_{L^2(\Omega_\varepsilon)}.\end{aligned}$$

Using Theorem 3.1 in the last term of the previous inequality, existence and uniqueness of the solution of problem (5) follow from an application of Lax-Milgram's lemma. □

Hereafter, we will always assume inequality (9) to hold. Moreover, by Theorem 3.3, we have that for any $\varepsilon > 0$ the energy functional

$$J_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} Du\mathring{T}^\varepsilon\cdot Du + \mathbb{L}Eu\cdot Eu\,dx - \int_{\Omega_\varepsilon} b^\varepsilon\cdot u\,dx \tag{10}$$

admits a unique minimizer among all displacements $u \in H^1_\flat(\Omega_\varepsilon;\mathbb{R}^3)$.

# 4 The rescaled problem

In order to study the behaviour of the energy functionals (10), as $\varepsilon \to 0$, following the idea of Ciarlet and Destuynder [4], we rescale the problem on a fixed domain. We consider the map $p_\varepsilon : \Omega \to \Omega_\varepsilon$ defined by

$$p_\varepsilon(y) := (\varepsilon y_1, \varepsilon y_2, y_3)$$

and introduce the rescaled energy $F_\varepsilon : H^1_\flat(\Omega;\mathbb{R}^3) \to \mathbb{R}$ by

$$F_\varepsilon(u) := \frac{1}{\varepsilon^2}J_\varepsilon(u\circ p_\varepsilon^{-1}).$$

Note that now the domain of the displacement $u$ is $\Omega$ and no longer $\Omega_\varepsilon$. We denote by $E^\varepsilon u := \mathrm{sym}(H^\varepsilon u)$ the rescaled strain, where

$$H^\varepsilon u := \left(\frac{D_1 u}{\varepsilon}, \frac{D_2 u}{\varepsilon}, D_3 u\right),$$

and $D_i u$ denotes the column vector of the partial derivatives of $u$ with respect to $y_i$, $i = 1, 2, 3$. Furthermore we also denote $W^\varepsilon u := \text{skw}(H^\varepsilon u)$, the skew-symmetric part of $H^\varepsilon u$. With $\mathring{T}$ as in Section 2, i.e., $\mathring{T} \in L^\infty(\Omega, \mathbb{R}^{3\times 3}_{\text{sym}})$ and satisfying (1), we set

$$\mathring{T}^\varepsilon = \varepsilon^2 \mathring{T} \circ p_\varepsilon^{-1}, \tag{11}$$

and denote by $\mathring{\tau}_m$ the smallest eigenvalue of $\mathring{T}$. We note that, under the change of variable $x = p_\varepsilon(y)$, the inequality (7) becomes

$$\int_\Omega |H^\varepsilon u|^2 dy \leq \frac{C_K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 dy \tag{12}$$

for every $u \in H^1_\flat(\Omega; \mathbb{R}^3)$ and for every $\varepsilon \in (0, 1]$.

Let us assume that

$$C_L > C_K |\mathring{\tau}_m|. \tag{13}$$

Note that by (11) this is equivalent to ask that the inequalities (9) be satisfied for any $\varepsilon$.

We consider the following splitting of the body forces $\tilde{b}$ introduced in Section 2:

$$\begin{gathered} \tilde{b}(y) = b_1(y) - \frac{m(y_3)}{I_O} y_2, \quad \tilde{b}(y) = b_2(y) + \frac{m(y_3)}{I_O} y_1, \\ \tilde{b}(y) = b_3(y), \end{gathered} \tag{14}$$

with $b = (b_1, b_2, b_3) \in L^2(\Omega; \mathbb{R}^3)$, $m \in L^2(0, \ell)$ and $I_O := \int_\omega (y_1^2 + y_2^2)\, dy_1\, dy_2$ the polar moment of inertia of the section $\omega$, and we define the sequence of body force densities $b^\varepsilon$, mentioned in Section 3, to be

$$\begin{gathered} b_1^\varepsilon \circ p_\varepsilon(y) = \varepsilon^2 b_1(y) - \varepsilon \frac{m(y_3)}{I_O} y_2, \quad b_2^\varepsilon \circ p_\varepsilon(y) = \varepsilon^2 b_2(y) + \varepsilon \frac{m(y_3)}{I_O} y_1, \\ b_3^\varepsilon \circ p_\varepsilon(y) = \varepsilon b_3(y). \end{gathered} \tag{15}$$

With these choices, and by performing the change of variable $x = p_\varepsilon(y)$, the rescaled energy $F_\varepsilon$ turns out to be

$$F_\varepsilon(u) = I_\varepsilon(u) - L_\varepsilon(u) \tag{16}$$

with

$$I_\varepsilon(u) := \frac{1}{2} \int_\Omega \mathbb{L} E^\varepsilon u \cdot E^\varepsilon u + \varepsilon^2 H^\varepsilon u \mathring{T} \cdot H^\varepsilon u \, dy, \tag{17}$$

$$L_\varepsilon(u) := \varepsilon^2 \int_\Omega b \cdot (u_1, u_2, \frac{u_3}{\varepsilon})\, dy - \varepsilon^2 \int_0^\ell m\, \vartheta^\varepsilon(u)\, dy_3, \tag{18}$$

and where we have set

$$\vartheta^\varepsilon(u)(y_3) := \frac{1}{I_O} \int_\omega \frac{y_1}{\varepsilon} u_2(y_1, y_2, y_3) - \frac{y_2}{\varepsilon} u_1(y_1, y_2, y_3)\, dy_1 dy_2. \tag{19}$$

We note that if $u \in L^2(\Omega; \mathbb{R}^3)$ then $\vartheta^\varepsilon(u) \in L^2(0, \ell)$. A similar statement holds if we replace $L^2$ with $H^1$.

**Remark 4.1.** It is worth notice that (13) is satisfied whenever the magnitude of the compressions "produced" by $\mathring{T}^\varepsilon$ are small enough. Together with the choice of scaling of $\mathring{T}^\varepsilon$ made in (11), assumption (13) ensures equi-coercivity of the sequence of the scaled energy functionals, as it will be proven in Lemma 6.1. That proof shows that assumption (11) is suggested by the scaled Korn's inequality (Theorem 5.1). Another thing to notice is that, by the composition with $p_\varepsilon$, the scaling (11) transforms the assumption $\mathrm{div}\mathring{T} = 0$ into

$$\varepsilon \mathring{T}^\varepsilon_{i1,1} + \varepsilon \mathring{T}^\varepsilon_{i2,2} + \mathring{T}^\varepsilon_{i3,3} = 0,$$

so, in general, $\mathring{T}^\varepsilon$ is not a divergence-free tensor field, but it is still symmetric and the normal trace at the boundary is still zero. It will be seen toward the end of the paper (see Remark 6.5) that if we further impose a divergence-free assumption on $\mathring{T}^\varepsilon$, then the average on the cross-section of $\mathring{T}$ vanishes and, as a consequence, the residual stress tensor would disappear in the limit problem, meaning that the divergence-free condition is, in some sense, not compatible with the chosen scaling.

# 5 Compactness lemmata

The following scaled Korn inequality will be used to prove compactness.

**Theorem 5.1.** *There exists a positive constant $K$, independent of $\varepsilon$, such that*

$$\int_\Omega |(u_1, u_2, \frac{u_3}{\varepsilon})|^2 + |H^\varepsilon u|^2 \, dy \leq \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy,$$

*for every $u \in H^1_\flat(\Omega; \mathbb{R}^3)$ and every $\varepsilon \in (0, 1]$.*

*Proof.* Setting $v = (u_1, u_2, u_3/\varepsilon)$ and noticing that $|E^\varepsilon u| \geq \varepsilon |Ev|$ and applying the standard Korn's inequality to $v$ on $\Omega$ (see, for instance, Oleinik, Shamaev and Yosifian [21], Theorem 2.7) we obtain that there exists a positive constant $K$ such that

$$\int_\Omega |(u_1, u_2, \frac{u_3}{\varepsilon})|^2 dy \leq \frac{K}{\varepsilon^2} \int_\Omega |E^\varepsilon u|^2 \, dy.$$

Using inequality (12) we conclude the proof. □

It will be useful also the following standard two-dimensional Korn's inequality:

$$\|w - \wp w\|^2_{H^1(\omega;\mathbb{R}^2)} \leq C \|Ew\|^2_{L^2(\omega;\mathbb{R}^{2\times 2})}, \tag{20}$$

for all $w \in H^1(\omega; \mathbb{R}^2)$, where $\wp$ denotes the projection of $L^2(\omega; \mathbb{R}^2)$ on the subspace

$$\mathcal{R}_2 = \{r \in L^2(\omega; \mathbb{R}^2) \,:\, \exists \, \varphi \in \mathbb{R}, \, c \in \mathbb{R}^2 \,:\, r_1(y) = -y_2\varphi + c_1, \, r_2(y) = y_1\varphi + c_2\}$$

of the infinitesimal rigid displacements on $\omega$ (see [21], Theorem 2.5 and Corollary 2.6, and [8]). It is easy to see that $\mathcal{R}_2$ is a closed subspace of $H^1(\omega; \mathbb{R}^2)$. Moreover, if $w \in L^2(\omega; \mathbb{R}^2)$ we have that

$$(\wp w)_\alpha = \mathcal{E}_{\beta\alpha} y_\beta \Big( \frac{1}{I_O} \int_\omega \mathcal{E}_{\gamma\delta} y_\gamma w_\delta \, dy_1 dy_2 \Big) + \frac{1}{|\omega|} \int_\omega w_\alpha \, dy_1 dy_2, \tag{21}$$

where $\mathcal{E}_{\alpha\beta}$ denotes the Ricci's symbol.

Let $H_{BN}(\Omega;\mathbb{R}^3) := \{v \in H^1_\flat(\Omega;\mathbb{R}^3) \;:\; (Ev)_{i\alpha} = 0 \text{ for } i = 1,2,3,\ \alpha = 1,2\}$ be the space of Bernoulli-Navier displacements on $\Omega$. It is well-known that it can be characterized also as follows (see Le Dret [18], Section 4.1)

$$\begin{aligned} H_{BN}(\Omega;\mathbb{R}^3) = \{\, v \in H^1_\flat(\Omega;\mathbb{R}^3) : \exists\, \xi_\alpha \in H^2_\flat(0,\ell), \exists\, \xi_3 \in H^1_\flat(0,\ell), \\ \text{s.t. } v_\alpha(y) = \xi_\alpha(y_3),\ v_3(y) = \xi_3(y_3) - y_\alpha \xi'_\alpha(y_3)\}, \end{aligned} \tag{22}$$

where

$$\begin{aligned} H^1_\flat(0,\ell) &= \{\xi \in H^1(0,\ell) \;:\; \xi(0) = 0\}, \\ H^2_\flat(0,\ell) &= \{\xi \in H^1(0,\ell) \;:\; \xi(0) = \xi'(0) = 0\}. \end{aligned}$$

In the remaining part of this section we assume $u^\varepsilon$ to be a sequence of functions in $H^1_\flat(\Omega;\mathbb{R}^3)$ such that

$$\|E^\varepsilon u^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \le C\varepsilon, \tag{23}$$

for some constant $C$ and for every $\varepsilon$. The next lemma summarizes and improves some results proven in [7, 8].

**Lemma 5.2.** *Let* (23) *hold for a sequence $u^\varepsilon \in H^1(\Omega;\mathbb{R}^3)$. Then*

1. *there exist a subsequence (not relabelled) and a couple of functions $v \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta \in L^2(\Omega)$ such that*

$$(u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon/\varepsilon) \rightharpoonup v \text{ in } H^1(\Omega;\mathbb{R}^3), \tag{24}$$

$$W^\varepsilon u^\varepsilon \rightharpoonup H(v,\vartheta) \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}), \tag{25}$$

$$H^\varepsilon u^\varepsilon \rightharpoonup H(v,\vartheta) \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}), \tag{26}$$

*where*

$$H(v,\vartheta) := \begin{pmatrix} 0 & -\vartheta & D_3 v_1 \\ \vartheta & 0 & D_3 v_2 \\ -D_3 v_1 & -D_3 v_2 & 0 \end{pmatrix}; \tag{27}$$

2. $\|\vartheta^\varepsilon(u^\varepsilon) + (W^\varepsilon u^\varepsilon)_{12}\|_{L^2(\Omega)} \le C\varepsilon$ *for some constant $C > 0$ and for every $\varepsilon \in (0,1]$;*

3. $\|\vartheta^\varepsilon(u^\varepsilon)\|_{L^2(\Omega)} \le \dfrac{C}{\varepsilon}\|E^\varepsilon u^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}$ *for some constant $C > 0$ and for every $\varepsilon \in (0,1]$;*

4. $\vartheta^\varepsilon(u^\varepsilon) \to \vartheta$ *in $L^2(\Omega)$ as $\varepsilon \to 0$; therefore $\vartheta$ does not depend on $y_1$ and $y_2$;*

5. $(H^\varepsilon u^\varepsilon)_{12} \to -\vartheta$ *in $L^2(\Omega)$, and $(H^\varepsilon u^\varepsilon)_{21} \to \vartheta$ in $L^2(\Omega)$ as $\varepsilon \to 0$;*

6. $\vartheta \in H^1_\flat(\Omega)$.

*Proof. 1.* It is convenient to set $v^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon/\varepsilon)$. Since $|E^\varepsilon u^\varepsilon| \ge \varepsilon |E v^\varepsilon|$, by (23), $Ev^\varepsilon$ is uniformly bounded in $L^2(\Omega;\mathbb{R}^{3\times 3})$ and by Korn's inequality $v^\varepsilon$ is uniformly bounded in $H^1(\Omega;\mathbb{R}^3)$. Then, there exist a $v \in H^1_\flat(\Omega;\mathbb{R}^3)$ and a subsequence (not relabelled) such that $v^\varepsilon \rightharpoonup v$ in $H^1(\Omega;\mathbb{R}^3)$. Again, it is easy to check that $|(E^\varepsilon u^\varepsilon)_{i\alpha}| \ge |(Ev^\varepsilon)_{i\alpha}|$, thus, using (23) we deduce that

$C\varepsilon \geq \|(Ev^\varepsilon)_{i\alpha}\|_{L^2(\Omega)}$ and consequently $(Ev)_{i\alpha} = 0$ for $i = 1,2,3$ and $\alpha = 1,2$. Hence $v \in H_{BN}(\Omega;\mathbb{R}^3)$. Using (23) and Theorem 5.1 we obtain that the sequence $H^\varepsilon u^\varepsilon$ is bounded in $L^2(\Omega;\mathbb{R}^{3\times 3})$ so that, up to subsequences, it weakly converges in $L^2(\Omega;\mathbb{R}^{3\times 3})$ to some $H \in L^2(\Omega;\mathbb{R}^{3\times 3})$. Since, from (23), $E^\varepsilon u^\varepsilon \to 0$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$, we have $W^\varepsilon u^\varepsilon \rightharpoonup H$ in $L^2(\Omega;\mathbb{R}^{3\times 3})$. In particular, $H$ is, almost everywhere, a skew-symmetric matrix. Since $(H^\varepsilon u^\varepsilon)_{13} = D_3 u_1^\varepsilon = D_3 v_1^\varepsilon$ and $(H^\varepsilon u^\varepsilon)_{23} = D_3 u_2^\varepsilon = D_3 v_2^\varepsilon$, we deduce that $(H)_{13} = D_3 v_1$ and $(H)_{23} = D_3 v_2$. We conclude the proof of *1* by denoting $(H)_{12} := -\vartheta$.

*2.* It is convenient to set $w^\varepsilon := (u_1^\varepsilon/\varepsilon, u_2^\varepsilon/\varepsilon)$. Then for almost $y_3 \in (0,\ell)$ and any $\varepsilon \in (0,1]$ we consider the projection of $w^\varepsilon(\cdot, y_3)$ on the space $\mathcal{R}_2$ of the infinitesimal rigid displacements on $\omega$. From the expression (21) of the projection $\wp$ and recalling the definition (19) of $\vartheta^\varepsilon$, we have

$$(\wp w^\varepsilon)_\alpha = \mathcal{E}_{\beta\alpha} y_\beta \vartheta^\varepsilon(u^\varepsilon) + \frac{1}{|\omega|}\int_\omega w_\alpha^\varepsilon \, dy_1 dy_2.$$

Since $(Ew^\varepsilon)_{11} = (E^\varepsilon u^\varepsilon)_{11}$, $(Ew^\varepsilon)_{12} = (E^\varepsilon u^\varepsilon)_{12}$ and $(Ew^\varepsilon)_{22} = (E^\varepsilon u^\varepsilon)_{22}$, we get $\|(Ew^\varepsilon)_{\alpha\beta}\|_{L^2(\Omega;\mathbb{R}^{2\times 2})} = \|(E^\varepsilon u^\varepsilon)_{\alpha\beta}\|_{L^2(\Omega;\mathbb{R}^{2\times 2})}$ for $\alpha, \beta = 1,2$. Then, writing (20) with $w^\varepsilon$ in place of $w$ and integrating on $(0,\ell)$, we deduce that

$$\int_0^\ell \|w^\varepsilon - \wp w^\varepsilon\|_{H^1(\omega;\mathbb{R}^2)} dy_3 \leq C \|E^\varepsilon u^\varepsilon\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}, \tag{28}$$

and taking also into account (23) we have

$$\|D_\alpha(w_\beta^\varepsilon - \wp w_\beta^\varepsilon)\|_{L^2(\Omega;\mathbb{R})} \leq C\varepsilon \quad (\alpha, \beta = 1,2).$$

Since $(W\wp w^\varepsilon)_{12} = -\vartheta^\varepsilon(u^\varepsilon)$ and $(Ww^\varepsilon)_{12} = (W^\varepsilon u^\varepsilon)_{12}$, we obtain *2* from the identity

$$\vartheta^\varepsilon(u^\varepsilon) = -(W\wp w^\varepsilon)_{12} = (W(w^\varepsilon - \wp w^\varepsilon))_{12} - (W^\varepsilon u^\varepsilon)_{12}. \tag{29}$$

From this last identity, using (28) and the scaled Korn's inequality Theorem 5.1 we get also claim *3*.

The weak convergence in *4* follows by taking the limit as $\varepsilon \to 0$ in (29) by using (25). Since $\vartheta^\varepsilon(u^\varepsilon)$ does not depend on $y_1$ and $y_2$, so does $\vartheta$. The strong convergence in *4* can be proven by adapting an argument of [8], Lemma 4.6, which consists in taking $\xi \in C_0^\infty(\omega)$ such that

$$\int_\omega \xi \, dy_1 \, dy_2 = -\frac{I_0}{2}$$

and define

$$\tilde{\vartheta}^\varepsilon = \frac{1}{I_0}\int_\omega \mathcal{E}_{\alpha\gamma}(D_\alpha \xi)\, w_\gamma^\varepsilon \, dy_1 \, dy_2.$$

Proceeding as in [8] we can prove that $\tilde{\vartheta}^\varepsilon - \vartheta^\varepsilon(u^\varepsilon) \to 0$ in $L^2(\Omega)$ and that $\tilde{\vartheta}^\varepsilon \rightharpoonup \vartheta$ in $H^1(\Omega)$, which implies the claimed strong convergence and *6*.

*5* follows from *2*, *4* and the fact that $(E^\varepsilon u^\varepsilon)_{12} \to 0$ in $L^2(\Omega)$. □

We now characterize the components of the limit strain $E$. Hereafter, we denote by

$$H_m^1(\omega) := \{v \in H^1(\omega) : \int_\omega v = 0\}$$

and

$$H_\wp^1(\omega;\mathbb{R}^2) := \{v \in H^1(\omega;\mathbb{R}^2) : \wp v = 0\}.$$

**Lemma 5.3.** *Let (23) hold for a sequence $u^\varepsilon$. Then there exist a subsequence, not relabeled, and a $E \in L^2(\Omega;\mathbb{R}^{3\times 3})$ such that*

$$\frac{E^\varepsilon u^\varepsilon}{\varepsilon} \rightharpoonup E \text{ in } L^2(\Omega).$$

*Moreover, there exist*

$$\varphi \in Q_1 := L^2((0,\ell);H^1_m(\omega)), \qquad w = (w_1,w_2) \in Q_2 := L^2((0,\ell);H^1_\wp(\omega;\mathbb{R}^2))$$

*such that*

$$E_{11} = (Ew)_{11}, \quad E_{22} = (Ew)_{22}, \quad E_{12} = (Ew)_{12}, \tag{30}$$

$$E_{13} = D_1\varphi - \frac{y_2}{2}D_3\vartheta, \qquad E_{23} = D_2\varphi + \frac{y_1}{2}D_3\vartheta, \tag{31}$$

$$E_{33} = D_3 v_3, \tag{32}$$

*where $v$ and $\vartheta$ have been defined in Lemma 5.2.*

*Proof.* Let $\bar{u}^\varepsilon$ be the vector whose components are the first two of $u^\varepsilon$, i.e. $\bar{u}^\varepsilon := (u_1^\varepsilon, u_2^\varepsilon)$. We have $(E\bar{u}^\varepsilon)_{\alpha\beta}/\varepsilon = (E^\varepsilon\bar{u}^\varepsilon)_{\alpha\beta}$, for $\alpha,\beta = 1,2$. Using (23) and integrating on $(0,\ell)$ the inequality (20) applied to the $\bar{u}^\varepsilon$, we find that

$$\|\frac{\bar{u}^\varepsilon - \wp\bar{u}^\varepsilon}{\varepsilon^2}\|_{L^2((0,\ell);H^1(\omega;\mathbb{R}^2))} \le C.$$

Hence, up to subsequences, $(\bar{u}^\varepsilon - \wp\bar{u}^\varepsilon)/\varepsilon^2 \rightharpoonup w$ in $L^2((0,\ell);H^1(\omega;\mathbb{R}^2))$ for some $w \in Q_2$. Moreover

$$\frac{(E^\varepsilon u^\varepsilon)_{\alpha\beta}}{\varepsilon} = \frac{E(\bar{u}^\varepsilon - \wp\bar{u}^\varepsilon)_{\alpha\beta}}{\varepsilon^2} \rightharpoonup (Ew)_{\alpha\beta} \text{ in } L^2(\Omega),$$

for $\alpha,\beta = 1,2$, and hence (30) has been proven. Equation (32) follows from (24). We now prove (31). Note that

$$D_3(W^\varepsilon u^\varepsilon)_{12} = D_2\big(\frac{(E^\varepsilon u^\varepsilon)_{13}}{\varepsilon}\big) - D_1\big(\frac{(E^\varepsilon u^\varepsilon)_{23}}{\varepsilon}\big),$$

in the sense of distributions. Hence for $\psi \in C_c^\infty(\Omega)$ we obtain

$$\int_\Omega (W^\varepsilon u^\varepsilon)_{12} D_3\psi\, dy = \int_\Omega \frac{(E^\varepsilon u^\varepsilon)_{13}}{\varepsilon} D_2\psi\, dy - \int_\Omega \frac{(E^\varepsilon u^\varepsilon)_{23}}{\varepsilon} D_1\psi\, dy.$$

Passing to the limit in the previous equality we find

$$\int_\Omega -\vartheta D_3\psi\, dy = \int_\Omega E_{13} D_2\psi\, dy - \int_\Omega E_{23} D_1\psi\, dy.$$

Thus $D_3\vartheta = -D_2E_{13} + D_1E_{23}$ in the sense of distributions. We can rewrite this equation as

$$D_2(E_{13} + \frac{y_2}{2}D_3\vartheta) = D_1(E_{23} - \frac{y_1}{2}D_3\vartheta)$$

in the sense of distributions. By the weak version of Poincaré's lemma (see Girault and Raviart [13], Theorem 2.9) there exists a function $\varphi \in Q_1$ such that

$$\begin{cases} E_{13} + \dfrac{y_2}{2}D_3\vartheta = D_1\varphi, \\ E_{23} - \dfrac{y_1}{2}D_3\vartheta = D_2\varphi, \end{cases}$$

which concludes the poof. □

From (30), (31) and (32) we have that the limit strain can be written as

$$
\begin{aligned}
E &= E(v,\vartheta,\varphi,w) \\
&:= \begin{pmatrix} (Ew)_{11} & (Ew)_{12} & D_1\varphi - \dfrac{y_2}{2}D_3\vartheta \\ (Ew)_{12} & (Ew)_{22} & D_2\varphi + \dfrac{y_1}{2}D_3\vartheta \\ D_1\varphi - \dfrac{y_2}{2}D_3\vartheta & D_2\varphi + \dfrac{y_1}{2}D_3\vartheta & D_3 v_3 \end{pmatrix}.
\end{aligned} \tag{33}
$$

# 6 The convergence result

Let $F_\varepsilon = I_\varepsilon - L_\varepsilon$ be the energy functionals defined by (16)-(18).

**Lemma 6.1.** *Let $u^\varepsilon$ be a sequence of functions in the space $H^1_\flat(\Omega;\mathbb{R}^3)$ such that*

$$\sup_\varepsilon \frac{F_\varepsilon(u^\varepsilon)}{\varepsilon^2} < +\infty.$$

*Then (23) holds for some constant $C>0$ and for every $\varepsilon$.*

*Proof.* It is convenient to set $v^\varepsilon := (u^\varepsilon_1, u^\varepsilon_2, u^\varepsilon_3/\varepsilon)$ and $R := C_L - C_K|\mathring{\tau}_m|$. By assumption (13), we have $R > 0$. With this notation and by using (2) and Lemma 3.2, for any $\varepsilon$ we find

$$
\begin{aligned}
\frac{1}{\varepsilon^2}F_\varepsilon(u^\varepsilon) &= \frac{1}{2}\int_\Omega \mathbb{L}\frac{E^\varepsilon u^\varepsilon}{\varepsilon}\cdot\frac{E^\varepsilon u^\varepsilon}{\varepsilon} + H^\varepsilon u^\varepsilon \mathring{T}\cdot H^\varepsilon u^\varepsilon\, dy + \\
&\quad - \int_\Omega b\cdot v^\varepsilon\, dy - \int_0^\ell m\vartheta^\varepsilon(u^\varepsilon)\, dy_3 \\
&\geq \frac{C_L}{2}\|\frac{E^\varepsilon u^\varepsilon}{\varepsilon}\|^2_{L^2(\Omega)} + \frac{\mathring{\tau}_m}{2}\|H^\varepsilon u^\varepsilon\|^2_{L^2(\Omega)} + \\
&\quad - \|b\|_{L^2(\Omega)}\|v^\varepsilon\|_{L^2(\Omega)} - \|m\|_{L^2(0,\ell)}\|\vartheta^\varepsilon(u^\varepsilon)\|_{L^2(0,\ell)} \\
&\geq \frac{R}{2}\|\frac{E^\varepsilon u^\varepsilon}{\varepsilon}\|^2_{L^2(\Omega)} - \|b\|_{L^2(\Omega)}\|v^\varepsilon\|_{L^2(\Omega)} - \|m\|_{L^2(0,\ell)}\|\vartheta^\varepsilon(u^\varepsilon)\|_{L^2(0,\ell)}
\end{aligned}
$$

where (12) has been used in the last inequality. From *3* of Lemma 5.2, the Young's inequality and Theorem 5.1 we obtain

$$
\begin{aligned}
\frac{1}{\varepsilon^2}F_\varepsilon(u^\varepsilon) \geq \frac{R}{2}\|\frac{E^\varepsilon u^\varepsilon}{\varepsilon}\|^2_{L^2(\Omega)} - \frac{1}{2C_1}\|b\|^2_{L^2(\Omega)} - \frac{C_1 K}{2}\|\frac{E^\varepsilon u^\varepsilon}{\varepsilon}\|^2_{L^2(\Omega)} + \\
- \frac{1}{2C_2}\|m\|^2_{L^2(0,\ell)} - \frac{C_2}{2}\|\frac{E^\varepsilon u^\varepsilon}{\varepsilon}\|^2_{L^2(\Omega)},
\end{aligned}
$$

where $C_1$ and $C_2$ are arbitrary positive constants. By choosing, for instance, $C_2 = R/2$ and $C_1 = R/(4K)$, we have

$$\frac{1}{\varepsilon^2}F_\varepsilon(u^\varepsilon) \geq \frac{R}{8}\|\frac{E^\varepsilon u^\varepsilon}{\varepsilon}\|^2_{L^2(\Omega)} - \frac{2K}{R}\|b\|^2_{L^2(\Omega)} - \frac{1}{R}\|m\|^2_{L^2(0,\ell)} \tag{34}$$

from which we get estimate (23). □

Lemma 6.1 and *1* of Lemma 5.2 imply that the family of functionals $(1/\varepsilon^2)F_\varepsilon$ is coercive in the space $H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R})$ with respect to the weak convergence of the sequence $q_\varepsilon(u^\varepsilon) := (u_1^\varepsilon, u_2^\varepsilon, u_3^\varepsilon/\varepsilon, (W^\varepsilon u^\varepsilon)_{12})$, uniformly with respect to $\varepsilon$. Hence, for any sequence $u^\varepsilon$ which is bounded in energy, that is $(1/\varepsilon^2)F_\varepsilon(u^\varepsilon)\le C$ for a suitable constant $C>0$, and satisfies the boundary conditions $u^\varepsilon = 0$ on $S(0)$, the corresponding sequence $q_\varepsilon(u^\varepsilon)$ is weakly relatively compact in $H^1(\Omega;\mathbb{R}^3)\times L^2(\Omega;\mathbb{R})$.

**Theorem 6.2** (Γ-convergence)**.** *Let $F : H^1_\flat(\Omega;\mathbb{R}^3)\times H^1_\flat(\Omega;\mathbb{R})\to\mathbb{R}\cup\{+\infty\}$ be defined by*

$$F(v,\vartheta) = I(v,\vartheta) - L(v,\vartheta) \tag{35}$$

*where*

$$\begin{aligned} I(v,\vartheta) &:= \frac{1}{2}\min_{\varphi\in Q_1, w\in Q_2}\Big\{\int_\Omega \mathbb{L}E(v,\vartheta,\varphi,w)\cdot E(v,\vartheta,\varphi,w)\,dy\Big\}+\\ &\quad+\frac{1}{2}\int_\Omega H(v,\vartheta)\mathring{T}\cdot H(v,\vartheta)\,dy, \qquad (36)\\ L(v,\vartheta) &:= \int_\Omega b\cdot v\,dy + \int_0^\ell m\vartheta\,dy_3, \end{aligned}$$

*if $v\in H_{BN}(\Omega;\mathbb{R}^3)$ and $D_1\vartheta = D_2\vartheta = 0$, and $+\infty$ otherwise, where $Q_1 = L^2((0,\ell);H^1_m(\omega))$, $Q_2 = L^2((0,\ell);H^1_\wp(\omega;\mathbb{R}^2))$ and $H(v,\vartheta)$ and $E(v,\vartheta,\varphi,w)$ are defined by (27) and (33). As $\varepsilon\to 0$, the sequence of functionals $(1/\varepsilon^2)F_\varepsilon$ Γ-converges to the functional $F$, in the following sense:*

1. *(liminf inequality) for every sequence of positive numbers $\varepsilon_k$ converging to 0 and for every sequence $\{u^k\}\subset H^1_\flat(\Omega;\mathbb{R}^3)$ such that*

$$(u_1^k, u_2^k, \frac{u_3^k}{\varepsilon_k}) \rightharpoonup v \text{ in } H^1(\Omega;\mathbb{R}^3), \qquad (W^{\varepsilon_k}u^k)_{12}\to -\vartheta \text{ in } L^2(\Omega), \tag{37}$$

*as $k\to\infty$, we have*

$$F(v,\vartheta)\le \liminf_{k\to+\infty}\frac{F_{\varepsilon_k}(u^k)}{\varepsilon_k^2};$$

2. *(recovery sequence) for every sequence of positive numbers $\varepsilon_k$ converging to 0 and for every $(v,\vartheta)\in H^1_\flat(\Omega;\mathbb{R}^3)\times H^1_\flat(\Omega;\mathbb{R})$ there exists a sequence $\{u^k\}\subset H^1_\flat(\Omega;\mathbb{R}^3)$ such that*

$$(u_1^k, u_2^k, \frac{u_3^k}{\varepsilon_k}) \rightharpoonup v \text{ in } H^1(\Omega;\mathbb{R}^3), \qquad (W^{\varepsilon_k}u^k)_{12}\to -\vartheta \text{ in } L^2(\Omega),$$

*as $k\to\infty$, and*

$$\limsup_{k\to+\infty}\frac{F_{\varepsilon_k}(u^k)}{\varepsilon_k^2}\le F(v,\vartheta).$$

*Proof. 1.* To prove the liminf inequality we can assume, possibly passing to subsequences, that

$$\liminf_{k\to+\infty}\frac{F_{\varepsilon_k}(u^k)}{\varepsilon_k^2} = \lim_{k\to+\infty}\frac{F_{\varepsilon_k}(u^k)}{\varepsilon_k^2} < +\infty.$$

Then Lemma 6.1 applies to the sequence $u^k$ and thereby the results of Lemma 5.2 and Lemma 5.3 hold. In particular, $v \in H_{BN}(\Omega;\mathbb{R}^3)$ and $D_1\vartheta = D_2\vartheta = 0$; moreover, besides (37) we have that

$$\frac{E^{\varepsilon_k}u^k}{\varepsilon_k} \rightharpoonup E(v,\vartheta,\varphi,w) \text{ (hence } E^{\varepsilon_k}u^k \to 0) \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}) \tag{38}$$

and

$$(H^{\varepsilon_k}u^k)_{12} \to -\vartheta \text{ in } L^2(\Omega), \quad (H^{\varepsilon_k}u^k)_{21} \to \vartheta \text{ in } L^2(\Omega). \tag{39}$$

Assumption (37) implies that

$$\frac{L_{\varepsilon_k}(u^k)}{\varepsilon_k^2} = \int_\Omega b\cdot(u_1^k,u_2^k,\frac{u_3^k}{\varepsilon_k})\,dy + \int_0^\ell m\vartheta^{\varepsilon_k}(u^k)\,dy_3 \to \int_\Omega b\cdot v\,dy + \int_0^\ell m\vartheta\,dy_3$$

and therefore we have only to prove that

$$\begin{aligned}\liminf_{k\to+\infty}\frac{I_{\varepsilon_k}(u^k)}{\varepsilon_k^2} \geq \frac{1}{2}\min_{\varphi\in Q_1,w\in Q_2}\Big\{\int_\Omega \mathbb{L}E(v,\vartheta,\varphi,w)\cdot E(v,\vartheta,\varphi,w)\,dy\Big\}+\\ +\frac{1}{2}\int_\Omega H(v,\vartheta)\mathring{T}\cdot H(v,\vartheta)\,dy.\end{aligned}$$

By setting $v^k := (u_1^k,u_2^k,u_3^k/\varepsilon_k)$, we can write

$$\begin{aligned}\frac{I_{\varepsilon_k}(u^k)}{\varepsilon_k^2} = \frac{1}{2}\int_\Omega\Big(&\mathbb{L}\frac{E^{\varepsilon_k}u^k}{\varepsilon_k}\cdot\frac{E^{\varepsilon_k}u^k}{\varepsilon_k} + \mathring{T}_{11}(D_1v_3^k)^2 + \mathring{T}_{33}\big((D_3v_1^k)^2+(D_3v_2^k)^2\big)+\\ &+\mathring{T}_{22}(D_2v_3^k)^2 + 2\mathring{T}_{12}(D_1v_3^k)(D_2v_3^k) + C_L\sum_{i=1}^3\sum_{\alpha=1}^2|(Ev^k)_{i\alpha}|^2+\\ &-C_L\sum_{i=1}^3\sum_{\alpha=1}^2|(Ev^k)_{i\alpha}|^2+\\ &+\mathring{T}_{11}\Big(\frac{(D_1v_1^k)^2}{\varepsilon_k^2}+\frac{(D_1v_2^k)^2}{\varepsilon_k^2}\Big)+\\ &+\mathring{T}_{33}\varepsilon_k^2(D_3v_3^k)^2 + \mathring{T}_{22}\Big(\frac{(D_2v_1^k)^2}{\varepsilon_k^2}+\frac{(D_2v_2^k)^2}{\varepsilon_k^2}\Big)+\\ &+2\mathring{T}_{12}\Big(\frac{D_1v_1^k}{\varepsilon_k}\frac{D_2v_1^k}{\varepsilon_k}+\frac{D_1v_2^k}{\varepsilon_k}\frac{D_2v_2^k}{\varepsilon_k}\Big)+\\ &+2\mathring{T}_{13}\Big(\frac{D_1v_1^k}{\varepsilon_k}D_3v_1^k+\frac{D_1v_2^k}{\varepsilon_k}D_3v_2^k+\varepsilon_k(D_1v_3^k)(D_3v_3^k)\Big)+\\ &+2\mathring{T}_{23}\Big(\frac{D_2v_1^k}{\varepsilon_k}D_3v_1^k+\frac{D_2v_2^k}{\varepsilon_k}D_3v_2^k+\varepsilon_k(D_2v_3^k)(D_3v_3^k)\Big)\Big)\,dy.\end{aligned} \tag{40}$$

Due to (37), (38) and (39), the last six lines in the inequality above converge to

$$\frac{1}{2}\int_\Omega(\mathring{T}_{11}+\mathring{T}_{22})\vartheta^2 + 2\mathring{T}_{13}\vartheta D_3v_2 - 2\mathring{T}_{23}\vartheta D_3v_1\,dy.$$

Indeed, in terms of $u^k$ the third line writes

$$-C_L\Big(\varepsilon^2\sum_{i=1}^2\sum_{\alpha=1}^2|(E^{\varepsilon_k}u^k)_{i\alpha}|^2 + \sum_{\alpha=1}^2|(E^{\varepsilon_k}u^k)_{3\alpha}|^2\Big)$$

which, as $k \to \infty$, goes to zero strongly in $L^1(\Omega)$ by (38). Analogously, the fourth line writes

$$\mathring{T}_{11}\Big(|(E^{\varepsilon_k}u^k)_{11}|^2 + \frac{|(H^{\varepsilon_k}u^k)_{21}|^2}{\varepsilon_k^2}\Big)$$

and strongly converges to $\mathring{T}_{11}\vartheta^2$ in $L^1(\Omega)$ due to (38) and (39), and so on.

Let us introduce now the following auxiliary quadratic functional

$$G(\gamma,\psi) := \frac{1}{2}\int_\Omega \mathbb{L}\gamma\cdot\gamma\,dy + C_L\sum_{i=1}^{3}\sum_{\alpha=1}^{2}|(E\psi)_{i\alpha}|^2 + \\ + \begin{pmatrix} 0 & 0 & D_1\psi_3 \\ 0 & 0 & D_3\psi_2 \\ D_3\psi_1 & D_2\psi_3 & 0 \end{pmatrix}\mathring{T}\cdot\begin{pmatrix} 0 & 0 & D_1\psi_3 \\ 0 & 0 & D_3\psi_2 \\ D_3\psi_1 & D_2\psi_3 & 0 \end{pmatrix}dy$$

with $\gamma \in L^2(\Omega;\mathbb{R}^{3\times 3})$ and $\psi \in W^{1,2}(\Omega;\mathbb{R}^3)$. Then the first two lines in (40) are given by $G(\frac{E^{\varepsilon_k}u^k}{\varepsilon_k}, v^k)$, and we have

$$\begin{aligned} G(\tfrac{E^{\varepsilon_k}u^k}{\varepsilon_k}, v^k) &= G(\tfrac{E^{\varepsilon_k}u^k}{\varepsilon_k} - E + E, v^k - v + v) \\ &\geq G(E,v) + G(\tfrac{E^{\varepsilon_k}u^k}{\varepsilon_k} - E, v) + G(E, v^k - v) \end{aligned} \tag{41}$$

since $G(\frac{E^{\varepsilon_k}u^k}{\varepsilon_k} - E, v^k - v) \geq 0$; indeed, by (2), the definitions of $C_L$ and $\mathring{\tau}_m$, Lemma 3.2, the standard Korn's inequality and assumption (13), we have

$$\begin{aligned} &G(\frac{E^{\varepsilon_k}u^k}{\varepsilon_k} - E, v^k - v) \geq \\ &\geq \frac{1}{2}\Big(C_L\|\frac{E^{\varepsilon_k}u^k}{\varepsilon_k} - E\|^2 - |\mathring{\tau}_m|\|D(v^k - v)\|^2 + C_L\sum_{i=1}^{3}\sum_{\alpha=1}^{2}\|(E(v^k-v))_{i\alpha}\|^2\Big) \\ &\geq \frac{1}{2}\Big(C_L\|\big(\frac{E^{\varepsilon_k}u^k}{\varepsilon_k} - E\big)_{33}\|^2 - |\mathring{\tau}_m|\|D(v^k - v)\|^2 + C_L\sum_{i=1}^{3}\sum_{\alpha=1}^{2}\|(E(v^k-v))_{i\alpha}\|^2\Big) \\ &= \frac{1}{2}\left(C_L\|E(v^k - v)\|^2 - |\mathring{\tau}_m|\|D(v^k-v)\|^2\right) \\ &= \frac{1}{2}(C_L - C_K|\mathring{\tau}_m|)\|E(v^k-v)\|^2 \geq 0. \end{aligned}$$

Hence, taking the limit as $k \to \infty$ in (40) and (41), we obtain

$$\begin{aligned} &\liminf_{k\to+\infty}\frac{I_{\varepsilon_k}(u^k)}{\varepsilon_k^2} \geq \\ &\quad\geq \frac{1}{2}\int_\Omega \big(\mathbb{L}E(v,\vartheta,\varphi,w)\cdot E(v,\vartheta,\varphi,w) + \mathring{T}_{11}(D_1v_3)^2 + \mathring{T}_{22}(D_2v_3)^2 + \\ &\qquad\qquad + \mathring{T}_{33}\big((D_3v_1)^2 + (D_3v_2)^2\big) + 2\mathring{T}_{12}(D_1v_3)(D_2v_3) + \\ &\qquad\qquad + (\mathring{T}_{11} + \mathring{T}_{22})\vartheta^2 + 2\mathring{T}_{13}\vartheta D_3v_2 - 2\mathring{T}_{23}\vartheta D_3v_1\big)\,dy \\ &\quad= \frac{1}{2}\int_\Omega \mathbb{L}E(v,\vartheta,\varphi,w)\cdot E(v,\vartheta,\varphi,w) + H(v,\vartheta)\mathring{T}\cdot H(v,\vartheta)\,dy \\ &\quad\geq \frac{1}{2}\inf_{\varphi\in Q_1, w\in Q_2}\Big\{\int_\Omega \mathbb{L}E(v,\vartheta,\varphi,w)\cdot E(v,\vartheta,\varphi,w)\,dy\Big\} + \\ &\qquad + \frac{1}{2}\int_\Omega H(v,\vartheta)\mathring{T}\cdot H(v,\vartheta)\,dy. \end{aligned}$$

The existence of the minimum in the previous inequality follows by a standard application of the direct method of the Calculus of Variations. Hence we have proven the liminf inequality.

*2.* Let us now find a recovery sequence. Let $F(v,\vartheta) < +\infty$, since otherwise there is nothing to prove. Then $v \in H_{BN}(\Omega;\mathbb{R}^3)$ and $\vartheta \in H^1_\flat(\Omega;\mathbb{R})$ with $D_1\vartheta = D_2\vartheta = 0$.

By (22), there exist $\xi_\alpha \in H^2(0,\ell)$ and $\xi_3 \in H^1(0,\ell)$ with $\xi_3(0) = \xi_\alpha(0) = \xi'_\alpha(0) = 0$ $(\alpha = 1,2)$, such that $v_\alpha(y) = \xi_\alpha(y_3)$, and $v_3(y) = \xi_3(y_3) - y_\alpha \xi'_\alpha(y_3)$. Let $\hat{w}$ and $\hat{\varphi}$ be the minimizers in the definition (36) of $F(v,\vartheta)$.

For any $\delta > 0$, we can find, by density, functions $\xi^\delta \in C^\infty(\Omega;\mathbb{R}^3)$, $\vartheta^\delta \in C^\infty(\Omega)$ with $D_1\vartheta^\delta = D_2\vartheta^\delta = 0$, $\hat{w}^\delta \in C^\infty(\Omega;\mathbb{R}^2)$ and $\hat{\varphi}^\delta \in C^\infty(\Omega)$, which are all equal to zero near by $y_3 = 0$ and such that

$$\xi^\delta_\alpha \to \xi_\alpha \text{ in } H^2(0,\ell), \quad \xi^\delta_3 \to \xi_3 \text{ in } H^1(0,\ell), \quad \vartheta^\delta \to \vartheta \text{ in } L^2(\Omega),$$
$$\hat{w}^\delta \to \hat{w} \text{ in } L^2((0,\ell),H^1(\omega;\mathbb{R}^2)), \quad \hat{\varphi}^\delta \to \hat{\varphi} \text{ in } L^2((0,\ell);H^1(\omega)).$$

For any $\delta > 0$, let $u^{\delta,k}$ be the sequence defined by

$$\begin{aligned} u^{\delta,k}_1 &:= \xi^\delta_1 - \varepsilon_k y_2 \vartheta^\delta + \varepsilon_k^2 \hat{w}^\delta_1, \\ u^{\delta,k}_2 &:= \xi^\delta_2 + \varepsilon_k y_1 \vartheta^\delta + \varepsilon_k^2 \hat{w}^\delta_2, \\ u^{\delta,k}_3 &:= \varepsilon_k(\xi^\delta_3 - y_1 \xi^{\delta\prime}_1 - y_2 \xi^{\delta\prime}_2) + 2\varepsilon_k^2 \hat{\varphi}^\delta. \end{aligned} \tag{42}$$

We have that $u^{\delta,k}$ is equal to zero in $y_3 = 0$ and it is easily checked that, setting $v^\delta_\alpha(y) := \xi^\delta_\alpha(y_3)$, and $v^\delta_3(y) := \xi^\delta_3(y_3) - y_\alpha \xi^{\delta\prime}_\alpha(y_3)$ and taking the limit as $k \to \infty$, we have

$$(u^{\delta,k}_1, u^{\delta,k}_2, \frac{u^{\delta,k}_3}{\varepsilon_k}) \to v^\delta \text{ in } H^1(\Omega;\mathbb{R}^3),$$

and

$$(W^{\varepsilon_k} u^{\delta,k})_{12} \to -\vartheta^\delta \text{ in } L^2(\Omega).$$

This implies that

$$\lim_{\delta\to 0^+} \lim_{k\to\infty} L_{\varepsilon_k}(u^{\delta,\varepsilon_k}) = \lim_{\delta\to 0^+} L(v^\delta,\vartheta^\delta) = L(v,\vartheta).$$

Moreover, it is easy to check that, taking the limit as $k \to \infty$ and then as $\delta \to 0^+$, we have

$$\frac{E^{\varepsilon_k} u^{\delta,k}}{\varepsilon_k} \to E(v^\delta,\vartheta^\delta,\hat{\varphi}^\delta,\hat{w}^\delta) \to E(v,\vartheta,\hat{\varphi},\hat{w}) \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}), \tag{43}$$

and

$$H^{\varepsilon_k} u^{\delta,k} \to H(v^\delta,\vartheta^\delta) \to H(v,\vartheta) \text{ in } L^2(\Omega;\mathbb{R}^{3\times 3}). \tag{44}$$

By (43) and (44) then we have

$$\begin{aligned} &\lim_{\delta\to 0^+} \lim_{k\to\infty} \frac{1}{\varepsilon_k^2} I_{\varepsilon_k}(u^{\delta,k}) = \\ &\quad = \frac{1}{2}\int_\Omega \mathbb{L}E(v,\vartheta,\hat{\varphi},\hat{w}) \cdot E(v,\vartheta,\hat{\varphi},\hat{w})\,dy + \frac{1}{2}\int_\Omega H(v,\vartheta)\mathring{T} \cdot H(v,\vartheta)\,dy. \end{aligned}$$

Thus, there exists a sequence of positive numbers $\delta_k \to 0$ such that the sequence $u^k := u^{\delta_k,k}$ is a recovery sequence.

□

For every $\varepsilon \in (0,1]$, let us denote by $\tilde{u}^\varepsilon$ the solution of the following minimization problem

$$\min\{F_\varepsilon(u) : u \in H^1_\flat(\Omega;\mathbb{R}^3)\}.$$

The next theorem follows from the $\Gamma$-convergence Theorem 6.2, the uniform coercivity of the sequence of the functionals $(1/\varepsilon^2)F_\varepsilon$ and the variational property of $\Gamma$-convergence (see for instance Dal Maso [5], Proposition 3.4).

**Theorem 6.3.** *The minimization problem for the* $\Gamma$*-limit functional* $F$ *defined in* (35)

$$\min\{F(v,\vartheta) : v \in H_{BN}(\Omega;\mathbb{R}^3),\ \vartheta \in H^1_\flat(0,\ell)\}$$

*admits a unique solution* $(\tilde{v},\tilde{\vartheta})$*. Moreover, as* $\varepsilon \to 0$*,*

1. $(\tilde{u}^\varepsilon_1, \tilde{u}^\varepsilon_2, \tilde{u}^\varepsilon_3/\varepsilon) \rightharpoonup \tilde{v}$ *in* $H^1(\Omega;\mathbb{R}^3)$*;*
2. $(W^\varepsilon \tilde{u}^\varepsilon)_{12} \to -\tilde{\vartheta}$ *in* $L^2(\Omega)$*;*
3. $(1/\varepsilon^2)F_\varepsilon(\tilde{u}^\varepsilon)$ *converges to* $F(\tilde{v},\tilde{\vartheta})$*.*

By using the Bernoulli-Navier structure of the domain, we show that the $\Gamma$-limit functional can be rewritten as a functional on $(0,\ell)$. To this aim, let $Q : (0,\ell)\times\mathbb{R}^4 \to [0,+\infty)$ be defined by

$$Q(y_3,a,b,c,d) := \min\Big\{ \int_\omega \mathbb{L}(y_3)\hat{E}\cdot\hat{E}\,dy_1dy_2\ \ : w \in H^1_\wp(\omega;\mathbb{R}^2),\ \varphi \in H^1_m(\omega)\Big\} \tag{45}$$

where

$$\hat{E} := \begin{pmatrix} (Ew)_{11} & (Ew)_{12} & D_1\varphi - \frac{y_2}{2}d \\ (Ew)_{12} & (Ew)_{22} & D_2\varphi + \frac{y_1}{2}d \\ D_1\varphi - \frac{y_2}{2}d & D_2\varphi + \frac{y_1}{2}d & c - ay_1 - by_2 \end{pmatrix}.$$

Let

$$\langle \mathring{T}\rangle := \int_\omega \begin{pmatrix} \mathring{T}_{11} & \mathring{T}_{12} \\ \mathring{T}_{21} & \mathring{T}_{22}\end{pmatrix} dy_1dy_2 \tag{46}$$

and

$$\langle b_i\rangle := \int_\omega b_i\,dy_1dy_2, \quad \langle b_3y_\alpha\rangle := \int_\omega b_3y_\alpha\,dy_1dy_2.$$

Let $BN(0,\ell) := H^2_\flat(0,\ell)\times H^2_\flat(0,\ell)\times H^1_\flat(0,\ell)$. Let $I_{1\mathrm{d}}, L_{1\mathrm{d}} : BN(0,\ell)\times H^1_\flat(0,\ell) \to \mathbb{R}$, be the functionals defined by

$$\begin{aligned} I_{1\mathrm{d}}(\xi,\vartheta) &:= \tfrac{1}{2}\textstyle\int_0^\ell Q(y_3,\xi_1'',\xi_2'',\xi_3',\vartheta') + \mathrm{tr}\langle\mathring{T}\rangle\,\vartheta^2 + \langle\mathring{T}\rangle(\xi_1',\xi_2')\cdot(\xi_1',\xi_2')\,dy_3 \\ L_{1\mathrm{d}}(\xi,\vartheta) &:= \textstyle\int_0^\ell \langle b_i\rangle\,\xi_i - \langle b_3y_\alpha\rangle\,\xi_\alpha' + m\vartheta\,dy_3 \end{aligned} \tag{47}$$

with the Einstein summation convention on $i = 1,2,3$ and $\alpha = 1,2$, and where tr denotes the trace. If $\xi_i$ are the Bernoulli-Navier components of $v$ (see (22)), also thanks to Lemma 2.1 it can be shown that

$$I(v,\vartheta) = I_{1\mathrm{d}}(\xi,\vartheta), \quad L(v,\vartheta) = L_{1\mathrm{d}}(\xi,\vartheta).$$

Within this framework, one has to solve the minimum problem (45) on the cross-section just like happens in Scardia [27] (Theorems 4.4 and 5.1) where a 1d linear model without residual stress is deduced starting from 3d nonlinear elasticity for a curved thin beam.

From (46) it would seem that only the averages of the in-section components of the residual stress influence the behavior of the beam but, in fact, the in-section components are in relation with the $\mathring{T}_{\alpha 3}$ components (see Lemma 2.1).

The next example shows that, in general, the contribution of the residual stress in the limit 1d model is non trivial.

**Example 6.4.** Let $f \in C_c^2(0, \ell)$ and $\omega = \{y \in \mathbb{R}^2 \ : \ |y| < 1\}$. Then the tensor field $\mathring{T}$ with components

$$\begin{aligned}
\mathring{T}_{11}(y) &= (y_1 y_2^3 + y_1^2 y_2^4) f''(y_3), \\
\mathring{T}_{22}(y) &= (y_1^3 y_2 + y_1^4 y_2^2) f''(y_3), \\
\mathring{T}_{12}(y) &= -(y_1^2 y_2^2 + y_1^3 y_2^3) f''(y_3), \\
\mathring{T}_{13}(y) &= (2y_1^2 y_2 - y_2^3 + 3y_1^3 y_2^2 - 2y_1 y_2^4) f'(y_3), \\
\mathring{T}_{23}(y) &= (2y_1 y_2^2 - y_1^3 + 3y_1^2 y_2^3 - 2y_1^4 y_2) f'(y_3), \\
\mathring{T}_{33}(y) &= (4y_1 y_2 + 2y_1^4 + 2y_2^4 - 18 y_1^2 y_2^2) f(y_3),
\end{aligned}$$

satisfies (1) on $\Omega = \omega \times (0, \ell)$. Moreover

$$\int_\omega \mathring{T}_{11}\, dy > 0, \quad \int_\omega \mathring{T}_{22}\, dy > 0, \quad \int_\omega \mathring{T}_{12}\, dy < 0,$$

on every section $y_3$ for which $f''(y_3) > 0$.

**Remark 6.5.** For $\mathring{T}^\varepsilon$ defined as in (11) the additional requirement $\operatorname{div}\mathring{T}^\varepsilon = 0$ would lead to

$$\mathring{T}_{i1,1} + \mathring{T}_{i2,2} + \varepsilon \mathring{T}_{i3,3} = 0$$

for $i = 1, 2, 3$. Since this holds for every $\varepsilon \in (0, 1]$, then we have

$$\mathring{T}_{i1,1} + \mathring{T}_{i2,2} = 0 \ \text{ and } \ \mathring{T}_{i3,3} = 0, \tag{48}$$

which are conditions much stronger than $\operatorname{div}\mathring{T} = 0$, see (1). The last equality implies that $\mathring{T}_{i3}$ is a function of $x_1$ and $x_2$ only and, in fact, $\mathring{T}_{i3} \in H^1(0, \ell; L^2(\omega))$. Since $\mathring{T}n = 0$ on $\partial\Omega$, we have that the trace of $\mathring{T}_{i3}$ on the bases $x_3 = 0$ vanishes and therefore we obtain

$$\mathring{T}_{i3} = 0, \quad (i = 1, 2, 3). \tag{49}$$

By *2* of Lemma 2.1 we immediately get

$$\int_\omega \mathring{T}_{\alpha\beta}\, dy_1 dy_2 = 0, \quad (\alpha, \beta = 1, 2),$$

and the residual stress tensor would completely disappear in the limit problem.

## 7 Remarks on the explicit computation of Q

This section is devoted to shed some light on the problem (45) defining $Q$.

For fixed $(a,b,c,d) \in \mathbb{R}^4$, $w \in H^1_\wp(\omega;\mathbb{R}^2)$ and $\varphi \in H^1_m(\omega)$, let

$$
\begin{aligned}
E^w &:= (Ew)_{\alpha\beta}\, \mathrm{e}_\alpha \odot \mathrm{e}_\beta,\\
E^\varphi &:= 2(D_1\varphi - \tfrac{y_2}{2}d)\,\mathrm{e}_1 \odot \mathrm{e}_3 + 2(D_2\varphi + \tfrac{y_1}{2}d)\,\mathrm{e}_2 \odot \mathrm{e}_3,\\
E^v &:= (c - ay_1 - by_2)\,\mathrm{e}_3 \otimes \mathrm{e}_3,
\end{aligned}
$$

where $(\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3)$ is the orthonormal basis associated to the axes $x_1$, $x_2$ and $x_3$. Above $\otimes$ denotes the dyadic product and $\odot$ is the associated symmetric product.

The minimizers $w \in H^1_\wp(\omega;\mathbb{R}^2)$ and $\varphi \in H^1_m(\omega)$ of problem (45) satisfy the Euler-Lagrange equations

$$
\begin{cases}
\displaystyle\int_\omega \mathbb{L}(y_3)(E^w + E^\varphi + E^v)\cdot E\eta\,dy_1dy_2 = 0 \quad \forall\,\eta \in H^1_\wp(\omega;\mathbb{R}^2),\\
\displaystyle\int_\omega \mathbb{L}(y_3)(E^w + E^\varphi + E^v)\cdot D\psi \odot \mathrm{e}_3\,dy_1dy_2 = 0 \quad \forall\,\psi \in H^1_m(\omega),
\end{cases}
\tag{50}
$$

where in computing $E\eta$ we consider $\eta$ as a three component vector field with third component equal to 0 and, similarly, we consider $\psi$ as a function of three variables.

We note that if

$$
\mathbb{L}_{3\alpha\beta\gamma} = 0 \quad (\alpha,\beta,\gamma = 1,2),
\tag{51}
$$

then

$$
\mathbb{L}(y_3)E^\varphi \cdot E\eta = \mathbb{L}(y_3)E^w \cdot D\psi \odot \mathrm{e}_3 = 0,
$$

hence (50) decouples into two separate problems, one for $w$ and one for $\varphi$. If, moreover, also

$$
\mathbb{L}_{333\gamma} = 0 \quad (\gamma = 1,2),
\tag{52}
$$

then

$$
\mathbb{L}(y_3)E^v \cdot D\psi \odot \mathrm{e}_3 = 0.
$$

Thus, under (51) and (52), problem (50) reduces to

$$
\begin{cases}
\displaystyle\int_\omega \mathbb{L}(y_3)(E^w + E^v)\cdot E\eta\,dy_1dy_2 = 0 \quad \forall\,\eta \in H^1_\wp(\omega;\mathbb{R}^2),\\
\displaystyle\int_\omega \mathbb{L}(y_3)E^\varphi\cdot D\psi \odot \mathrm{e}_3\,dy_1dy_2 = 0 \quad \forall\,\psi \in H^1_m(\omega),
\end{cases}
\tag{53}
$$

from which we deduce that the unknown $\varphi$ depends only on the constant $d$, i.e. on $D_3\vartheta$, and not on $a$, $b$, $c$.

Equations (51) and (52) are satisfied for a monoclinic material with uniform axis of symmetry $\mathrm{e}_3$. If we impose the same kind of symmetry on the residual stress, i.e.

$$
\mathring{T}_0 = R\mathring{T}R^T
\tag{54}
$$

for every rotation $R$ in the monoclinic symmetry group, we would deduce $\mathring{T}_{i3} = 0$, $i = 1,2,3$, see Hoger [16, (5.9)]. From Lemma 2.1 we then deduce that

$\langle \mathring{T}_{12} \rangle = 0$ which would imply that the Γ-limit does not depend on the residual stress.

We note that the symmetry group $\mathscr{G}$ of a material is contained in the orthogonal group only if the body is in its undistorted reference configuration. In any other reference, $\kappa$, the symmetry group will be $F\mathscr{G}F^{-1}$ where $F$ is the gradient of the mapping from the undistorted reference configuration to $\kappa$ (see Truesdell [31]). Thus, we believe that it is restrictive to assume the symmetry group to be contained in the orthogonal group (see also [24]) and hence also (54).

We conclude by looking at the case of a slender rod made of isotropic, homogeneous material with a stress-free reference configuration.

In this case we have

$$\mathbb{L}E(v,\vartheta,\varphi,w)\cdot E(v,\vartheta,\varphi,w) = 2\mu|E(v,\vartheta,\varphi,w)|^2 + \lambda|\mathrm{tr}(E(v,\vartheta,\varphi,w))|^2,$$

where $\mu > 0$ and $\lambda \geq 0$ are the Lamé moduli of the material.

With the isotropy symmetry condition the unknowns $w$ and $\varphi$ satisfy problem (53) which rewrites as

$$\begin{cases} \displaystyle\int_\omega \big((2\mu E^w + \lambda(\mathrm{tr}E^w + c - ay_1 - by_2)I\big)\cdot D\eta\, dy_1 dy_2 = 0 \quad \forall\, \eta \in H^1_\wp(\omega;\mathbb{R}^2), \\ \displaystyle\int_\omega D\varphi\cdot D\psi + \frac{d}{2}(-y_2,y_1)\cdot D\psi\, d_1 dy_2 = 0 \quad \forall\, \psi \in H^1_m(\omega). \end{cases} \tag{55}$$

It can be checked that the solution of $(55)_1$ is

$$\begin{aligned} w_1 &= -\nu\big(cy_1 - a\frac{y_1^2+y_2^2}{2} - by_1y_2\big) - k_3y_2 + k_1, \\ w_2 &= -\nu\big(cy_2 - b\frac{y_1^2+y_2^2}{2} - ay_1y_2\big) + k_3y_2 + k_2, \end{aligned}$$

where $k_1$, $k_2$ and $k_3$ are found by imposing that $\wp w = 0$, and where $\nu := \lambda/(2\lambda+2\mu)$ denotes the Poisson's ratio. The solution of $(55)_2$ is given by

$$\varphi = \frac{d}{2}\varphi_T,$$

where $\varphi_T$ is the *torsion function*

$$\begin{cases} \triangle\varphi_T = 0 & \text{in } \omega, \\ D\varphi_T\cdot n = -(-y_2,y_1)\cdot n & \text{on } \partial\omega. \end{cases}$$

With $w$ and $\varphi$ as above we can explicitly compute the function $Q$ given by (45):

$$\begin{aligned} Q(y_3,a,b,c,d) &= \int_\omega (c - ay_1 - by_2)^2E + d^2\mu|D\varphi_T + (-y_2,y_1)|^2\, dy_1dy_2 \\ &= EAc^2 + EJ_2a^2 + EJ_1b^2 + d^2\mu\int_\omega |D\psi_T|^2\, dy_1dy_2, \end{aligned}$$

where $E := (2\mu^2 + 3\lambda\mu)/(\mu+\lambda)$ denotes the Young modulus, $A$, $J_1$ and $J_2$ are the area and the principal moments of inertia of the cross-section $\omega$, and $\psi_T$ is the so-called *Prandtl stress function* defined by

$$\begin{cases} \triangle\psi_T = -2, \\ \psi_T \in H^1_0(\omega). \end{cases}$$

Hence, the energy $I_{1d}$ given by (47) reduces to

$$I_{1d}(\xi,\vartheta) = \frac{1}{2}\int_0^\ell EA{\xi_3'}^2 + EJ_2{\xi_1''}^2 + EJ_1{\xi_2''}^2 + \mu J_T{\vartheta'}^2\, dy_3,$$

where

$$J_T := \int_\omega |D\psi_T|^2\, dy_1 dy_2,$$

and it coincides with the energy obtained by Anzellotti, Baldo and Percivale [3] (in the case of a circular cross-section), Percivale [26], and also in [7].

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