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An age-structured population dynamics model for several species with finite life-span

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Abstract

In this paper we propose a nonlinear model to describe the dynamics of more interacting populations or species which generalizes the Gurtin-MacCamy one and we present some numerical results.

1 Introduction

In this paper we propose a nonlinear model to describe the dynamics of more interacting populations or species which generalizes the Gurtin-MacCamy one. The idea is that the basic dynamics of each populations is well-describe by the Gurtin-MacCamy model, but we also want to take into account how the vital rates can be influenced by the presence of other populations or species. This report is the result of the master's thesis [19]. In the first part we give the basic results on existence and uniqueness of solutions, while the second part is devoted on some numerical experiments.

2 The Model

Let us consider a population consisting of d species with age structure. The dynamics of such population is described at time t by the following vector-valued age density function:

$$p(t, a) \in \mathbb{R}^d,\ t \geq 0,\ a \in [0, a_\dagger],$$

where $a_\dagger$ denotes the maximum age of survival among all the species. To consider the biologically more realistic situation in which all the species of the population have a finite life-span, we assume that maximum age $a_\dagger$ is finite.

The equations governing the dynamics of the population generalize the nonlinear Gurtin-Mac Camy model (see [13]) as follows

$$\frac{\partial p}{\partial t}(t,a) + \frac{\partial p}{\partial a}(t,a) + \mu(t,a,S(t))p(t,a) = 0,\ a \in [0,a_\dagger),\ t > 0, \tag{1}$$

$$p(t,0) = \int_0^{a_\dagger} \beta(t,a,S(t))p(t,a)da,\ t \geq 0, \tag{2}$$

$$S(t) = \int_0^{a_\dagger} \gamma(a)p(t,a)da,\ t \geq 0, \tag{3}$$

where

$$\gamma : [0,a_\dagger] \to \mathbb{R}^{n\times d},$$

and $S(t) \in \mathbb{R}^n$. The n components of $S(t)$ are called *sizes* and represent different ways of weighting the age distributions.

Concerning the vital rates, we assume that the diagonal elements of the mortality matrix $\mu(t,a,S) \in \mathbb{R}^{d\times d}$, describing the mortality of the species, are separable into two terms, i.e.

$$\mu_{ii}(t,a,S) = \mu_i^{(0)}(a) + \mu_i^{(1)}(t,a,S),\ i = 1,...,d. \tag{4}$$

The functions $\mu_i^{(0)}$, $i = 1,...,d$, in (4) denote the "natural" death rate of the i-th specie and are unbounded to take into account a maximum age $a_\dagger$, $i.e.$

$$\int_0^{a_\dagger} \mu_i^{(0)}(a)da = +\infty,\ i = 1,...,d. \tag{5}$$

By introducing the diagonal matrix

$$\mu^{(0)}(a) = diag(\mu_1^{(0)}(a),...\mu_d^{(0)}(a)), \tag{6}$$

the mortality matrix μ can be also represented as follows

$$\mu(t,a,S) = \mu^{(0)}(a) + \mu^{(1)}(t,a,S), \tag{7}$$

where $\mu^{(0)}(a) \in \mathbb{R}^{d\times d}$ is the *intrinsic* mortality matrix and $\mu^{(1)}(t,a,S) \in \mathbb{R}^{d\times d}$ is the *external* mortality matrix.

Moreover we assume that either the external mortality matrix or the fertility matrix $\beta(t,a,S) \in \mathbb{R}^{d\times d}$ include seasonality through the dependence on the time t and resource competition through the dependence on S.

Let us introduce some notations. In the vector space $\mathbb{R}^d$ we will consider the norm $|x| = \sum_{i=1}^d |x_i|,\ x \in \mathbb{R}^d$, which is more natural for population problems, while the linear space of $d \times d$ matrices is equipped with the induced norm $|M| = \max_{|x|=1} |Mx|,\ M \in \mathbb{R}^{d\times d}$.

Since

$$n(t) = \int_0^{a_\dagger} p(t,a)da$$

is the vector-valued function whose components give the total number of individuals of each species in the population at time t, we should have $p(t,\cdot) \in X = L^1([0,a_\dagger);\mathbb{R}^d)$, i.e. the space of all (equivalent classes of) $\mathbb{R}^d$-valued measurable functions φ defined and absolutely integrable in $[0,a_\dagger)$ equipped with the norm

$$\|\varphi\|_X := \int_0^{a_\dagger} |\varphi(a)|da.$$

Moreover $\mathbb{R}^d_+$ and X_+ denote respectively cone of vectors and functions with nonnegative components.

Let $T > 0$ and define the Banach space $\mathcal{C}_T = \mathcal{C}([0,T], X)$ with the norm

$$||p||_T = \sup_{0 \le t \le T} ||p(t)||_X, \quad p \in \mathcal{C}_T.$$

Each element of $\mathcal{C}_T$ can be identified in a natural way with an element of $L^1([0,T] \times [0,a_\dagger), \mathbb{R}^d)$ and we will use the same symbol p to denote both elements, i.e.

$$p(t)(a) = p(t,a), \quad a.e.\ a \in [0,a_\dagger),\ t \in [0,T],$$

(see [27]). In the sequel we will consider also the spaces $L^{1,\infty}([0,a_\dagger); \mathbb{R}^{m \times n})$, i.e. the space of all (equivalent classes of) $\mathbb{R}^{m \times n}$-valued measurable functions M defined in $[0,a_\dagger)$ equipped with the norm

$$||M|| = |||M|||_{1,\infty}. \tag{8}$$

In the contest of populations only positive solutions have significance and therefore, given a nonnegative funcion $p_0 \in X_+$, we are interested to provide a result on the existence and uniqueness of a positive solution of the problem (1)-(2)-(3) under the initial condition

$$p(0,a) = p_0(a), \quad a.e.\ a \in [0,a_\dagger). \tag{9}$$

For all $T > 0$, we assume that

(H1) the matrix
$$\gamma : [0,a_\dagger] \to \mathbb{R}^{n \times d},$$
belongs to $L^\infty([0,a_\dagger]; \mathbb{R}^{n \times d})$ and it is a.e. nonnegative;

(H2) the fertility matrix
$$\beta : [0,T] \times [0,a_\dagger] \times \mathbb{R}^n \to \mathbb{R}^{d \times d}$$
is such that

- $\beta(t,a,S)$ is nonnegative a.e. on $[0,T] \times [0,a_\dagger] \times \mathbb{R}^n$,
- $t \to \beta(t,\cdot,\cdot)$ is continuous on $[0,T]$,
- $a \to \beta(\cdot,a,\cdot)$ belongs to $L^\infty([0,a_\dagger]; \mathbb{R}^{d \times d})$,
- $S \to \beta(\cdot,\cdot,S)$ is locally Lipschitz continuous, i.e. there exists an increasing function $K_\beta : [0,+\infty) \to [0,+\infty)$ such that, if $S, \bar{S} \in \mathbb{R}^n$,$|S|, |\bar{S}| \le \rho$, then
$$|\beta(t,a,S) - \beta(t,a,\bar{S})| \le K_\beta(\rho)|S - \bar{S}|,\ a.e\ a \in [0,a_\dagger], t \in [0,T],$$
- there exists an increasing function $\beta^* : [0,+\infty) \to [0,+\infty)$ such that
$$|\beta(t,a,S)| \le \beta^*(|S|), a.e.a \in [0,a_\dagger], t \in [0,T], S \in \mathbb{R}^n;$$

(H3) the mortality matrix
$$\mu : [0,T] \times [0,a_\dagger] \times \mathbb{R}^n \to \mathbb{R}^{d \times d}$$
is such that

o (4) holds and thus it an be represented as the sum of the internal mortality matrix $\mu^{(0)}$ and the external mortality $\mu^{(1)}$ as in (7),

o the functions $\mu_i^{(0)} \in L^1_{\text{loc}}([0,a_\dagger);\mathbb{R}), i = 1,...,d$, are a.e. nonnegative and moreover

$$\int_0^{a_\dagger} \mu_i^{(0)}(a)da = +\infty, i = 1,...d, \tag{10}$$

o the functions $\mu_i^{(1)}(\cdot,a,\cdot)$ belongs to $L^1([0,a_\dagger],\mathbb{R}) i = 1,...,d$

o $t \to \mu(t,\cdot,\cdot)$ is continuous on $[0,T]$,

o there is an increasing function $K_1 : [0,+\infty) \to [0,+\infty)$ such that for all $i,j, i \neq j$

$$|\mu_{i,j}(t,\bar{a},S) - \mu_{i,j}(t,a,S)| \leq K_1(|S|)|a-\bar{a}|, t \in [0,T],$$

o $S \to \mu_1(\cdot,\cdot,S)$ is locally Lipschitz continuous, i.e. there exists an increasing function $K_\mu : [0,+\infty) \to [0,+\infty)$ such that, if $S, \bar{S} \in \mathbb{R}^n$,$|S|, |\bar{S}| \leq \rho$, then

$$|\mu(t,a,S) - \mu(t,a,\bar{S})| = \leq K_\mu(\rho)|S-\bar{S}|,\ a.e\ a \in [0,a_\dagger], t \in [0,T],$$

o there exists an increasing function $\mu^* : [0,+\infty) \to [0,+\infty)$

$$|\mu^{(1)}(t,a,S)| \leq \mu^*(|S|),\ a.e\ a \in [0,a_\dagger], t \in [0,T],$$

(H4) for all $\rho > 0$, given $S \in \mathbb{R}^n, |S| \leq \rho$ and $a.e.\ a \in [0,a_\dagger), t \in [0,T]$

$$-\mu_i^{(1)}(t,a,S) + \textstyle\sum_{j=1,j\neq i}^d |\mu_{ij}(t,a,S)| \leq \mu_i^{(0)}(a),\ i = 1,...d,$$

$$\mu_{ij}(t,a,S) \leq 0,\ i,j = 1,...,d, i \neq j;$$

(H5) the initial function p_0 belongs to X_+.

Even if our aim is to study the model (1)-(2)-(3), it is useful to reformulate it in a more general form, which is applicable to several population problems, as in [27]. This allow us to use a compact notation and to better compare the results with those already developed in the literature.
By introducing the mapping $\mathcal{G} : [0,T] \times X \to L^1_{\text{loc}}$ (*aging function*)

$$\mathcal{G}(t,\varphi)(a) = -\mu(t,a,\mathcal{S}\varphi)\varphi(a),\ t \in [0,T],\ a.e.\ a \in (0,a_\dagger) \tag{11}$$

and the mapping $\mathcal{F} : [0,T] \times X \to \mathbb{R}^d$ (*birth function*)

$$\mathcal{F}(t,\varphi) = \int_0^{a_\dagger} \beta(t,a\mathcal{S}\varphi)\varphi(a)da, \tag{12}$$

where $\mathcal{S} : X \to \mathbb{R}^n$

$$\mathcal{S}\varphi = \int_0^{a_\dagger} \gamma(a)\varphi(a)da, \tag{13}$$

we can reformulate (1)-(2)-(3)-(9) as follows: find a nonnegative (strong) solution $p \in \mathcal{C}_T$, satisfying the initial condition (9) and

$$\mathcal{D}p(t,a) = \mathcal{G}(t,p(t,\cdot))(a),\ t \in (0,T),\ a.e.\ a \in (0,a_\dagger), \tag{14}$$

$$p(t,0) = b(t),\ t \in [0,T], \tag{15}$$

supplemented by the condition

$$b(t) = \mathcal{F}(t, p(t,\cdot)),\ t \in [0,T], \tag{16}$$

where $\mathcal{D}$ is the differential operator

$$\mathcal{D}p(t,a) =: \lim_{h\to 0} \frac{p(t+h,a+h) - p(t,a)}{h}.$$

Remark 1 *Observe that the aging and the birth functions in the general model studied in [27] don't depend on t and so the effect of stagionality is not included. Moreover here we consider the case in which the aging function is separable in a linear part, i.e.*

$$\mathcal{L}(\varphi)(a) = -\mu^{(0)}(a)\varphi(a), \tag{17}$$

which unbounded and in a nonlinear one, i.e.

$$\mathcal{N}(t,\varphi)(a) = -\mu^{(1)}(t,a,\mathcal{S}\varphi)\varphi(a). \tag{18}$$

Remark 2 *In the case $d = 1$, the assumptions (H1)—(H5) reduce to those considered in the book [17], where the author studies separable models with external mortality depending only on the sizes.*

Remark 3 *Observe that the assumptions (H1)-(H2) ensure that the birth function (12) is continuous with respect to t and lipschitz continuous with respect to φ on bounded sets of X uniformly with respect to $t \in [0,T]$. Specifically, we have that there exists a positive increasing function $K_{\mathcal{F}}(\rho) > 0$ such that for all $\rho > 0$ if $\varphi, \bar{\varphi} \in X$, $|\varphi|, |\bar{\varphi}| \leq \rho$, then*

$$|\mathcal{F}(t,\varphi) - \mathcal{F}(t,\bar{\varphi})| \leq K_{\mathcal{F}}(\rho)||\varphi - \bar{\varphi}||_X,\ t \in [0,T]. \tag{19}$$

Moreover we can also easily obtain the following property

$$\mathcal{F}(t,\varphi) \in \mathbb{R}^d_+,\ t \in [0,T],\ \varphi \in X_+, \tag{20}$$

which is relevant for finding positive solutions.

Remark 4 *Observe that the assumptions (H1)-(H3) ensure that the aging function (11) is continuous with respect to t and that the non linear part (18) is lipschitz continuous with respect to φ on bounded sets of X uniformly with respect to $t \in [0,T]$.*

In order to prove the well-posedness of the problem (14)-(15)-(16)-(9) with birth and aging functions given by (12) and (11) respectively, we will use the method of characteristics with a fixed point argument. This is a standard approach (see [9], [27], [17]).

Define, for fixed $t, a > 0$, $c := a - t$. For $-T < c < a_\dagger$, by (14) we obtain that the *cohort* function

$$P_c(s) := p(s, s+c), \quad t_c := \max\{0, -c\} \leq s \leq T_c = \max\{T, a_\dagger - c\},$$

satisfies

$$\frac{d}{ds}P_c(s) = \mathcal{G}(s, p(s,\cdot))(s+c),\ a.e.\ s \in [t_c, T_c). \tag{21}$$

For the aging function $\mathcal{G}$ having the form (11), we can define for $-T < c < a_\dagger, q \in \mathcal{C}_T$, the family of matrices

$$M(s; c, q) := \quad -\mu(s, s+c, \mathcal{S}q(s, \cdot)), \quad a.e.\ s \in [t_c, T_c). \tag{22}$$

If, for $-T < c < a_\dagger$ and $q \in \mathcal{C}_T$, we can define the family of evolution operators $\{\mathcal{U}(s, \sigma; c, q) \in \mathbb{R}^{d\times d} |\ t_c \le \sigma \le s \le T_c\}$ associated with

$$\frac{d}{ds}u(s) = M(s; c, q)u(t), a.e. s \in [\sigma, T_c), \quad u(\sigma) = x \tag{23}$$

i.e. $u(s) = \mathcal{U}(s, \sigma; c, q)x, x \in \mathbb{R}^d$, then the solution of (14)-(15)-(16)-(9) is given by

$$p(t, a) = \begin{cases} \mathcal{U}(t, t-a; a-t, p)b(t-a; p), & a.e.\ a \in (0, t) \\ \mathcal{U}(t, 0; a-t, p)p_0(a-t), & a.e.\ a \in (t, a_\dagger) \end{cases}.$$

where, for a given $q \in \mathcal{C}_T$, $b(t; q)$ denotes the solution of the the *renewal equation*

$$\begin{aligned} v(t) = \quad & \int_0^t \beta(t, a, \mathcal{S}q(t, \cdot))\mathcal{U}(t, t-a; a-t, q)v(t-a)da + \\ & \int_t^{a_\dagger} \beta(t, a, \mathcal{S}q(t, \cdot))\mathcal{U}(t, 0; a-t, q)p_0(t-a)da,\ t \in [0, T]. \end{aligned} \tag{24}$$

We give now some results on the solutions either of (23) or (24).

In the scalar case, the evolution operator associated with (23) is

$$\mathcal{U}(s, \sigma; c, q) = e^{\int_\sigma^s M(t; c, q)dt}, \quad t_c \le \sigma \le s \le T_c,$$

in the general case we have

Lemma 5 *Let $T > 0$. Under the assumptions (H1), (H3)-H(4) given $-T < c < a_\dagger$ and $q \in \mathcal{C}_T$, the family of evolution operators $\{\mathcal{U}(s, \sigma; c, q) \in \mathbb{R}^{d\times d} |\ t_c \le \sigma \le s \le T_c\}$ is well defined and for $t_c \le \sigma \le s < T_c$ the following properties hold*

(i) $\frac{\partial}{\partial s}\mathcal{U}(s, \sigma; c, q) = M(s; c, q)\mathcal{U}(s, \sigma; c, q)$;

(ii) $\frac{\partial}{\partial \sigma}\mathcal{U}(s, \sigma; c, q) = -\mathcal{U}(s, \sigma; c, q)M(s; c, q)$;

(iii) $|\mathcal{U}(s, \sigma; c, q)| \le 1$;

(iv) $\mathcal{U}(s, \sigma; c, q)x \ge 0$, *for* $x \ge 0$.

Proof. By introducing the matrices $R(s, \sigma) = diag(exp^{\int_\sigma^s M_{ii}(t; c; q)tdt})$, $\sigma \le s < T_c$, and $\tilde{M} = M - diag(M_{ii})$, we can represent the solution of (23) as $u(s) = R(s, \sigma)w(s)$, where w satisfies

$$R(t, \sigma)\frac{d}{dt}w(t) = \tilde{M}(t; c, q)(t)R(t, \sigma)w(t),\ t \in [\sigma, s], \quad w(\sigma) = x. \tag{25}$$

The function $t \to R(t, \sigma)^{-1}\tilde{M}(t; c, q)R(t, \sigma)$ is continuous and there exists a unique continuous solution of (25) on the interval $[\sigma, s]$ which is differentiable almost everywhere on (σ, s). Thus the evolution operator $\mathcal{U}(s, \sigma; c, q)$ is well-defined and (i)-(ii) are true. By virtue of (H4) we have that logarithmic norm of the matrix M is nonnegative and (iii) follows. Moreover the solution is positive by standard results (see [15]). ■

Remark 6 *Observe that we can assume $\mathcal{U}(T_c, \sigma; c, q) = 0$ extending the definition of the evolution operator.*

Remark 7 *Observe that in the scalar case or in the case that all the species of the population have the same intrinsic mortality, i.e. $\mu_i^{(0)}(a) = \mu(0)(a)$, as in the epidemiology models, we have that*

$$R(t, \sigma) = exp^{-\int_\sigma^s \mu^{(0)}(t+c)dt})diag(exp^{-\int_\sigma^s \mu_i^{(1)}(t,t+c,\mathcal{S}q(t,\cdot))dt}),$$

*and the component $exp^{-\int_\sigma^s \mu^{(0)}(t+c)dt}$ can be simplified in (**??**).*

Lemma 8 *Let $T > 0$. Under the assumptions (H1),(H3)-H(4) for $-T < c < a_\dagger$ and $q \in \mathcal{C}_T$, the family of evolution operators $\{\mathcal{U}(s, \sigma; c, q) \in \mathbb{R}^{d\times d} | \; t_c \leq \sigma \leq s \leq T_c\}$ satisfies for all $q, \tilde{q} \in \mathcal{C}_T$, $||q||_T, ||\tilde{q}||_T \leq \rho$*

$$|\mathcal{U}(s, \sigma; c, q) - \mathcal{U}(s, \sigma; c, \tilde{q})| \leq ||q - \tilde{q}||_T K_1(\gamma^+ \rho)\gamma^+ T.$$

Proof. The property follow from lemma 5 and the fact

$$\frac{\partial}{\partial \tau}\mathcal{U}(s, \tau; c, q)\mathcal{U}(\tau, \sigma; c, \tilde{q}) = -\mathcal{U}(s, \tau; c, q)[M(\tau; c, q) - M(\tau; c, \tilde{q}))\mathcal{U}(\tau, \sigma; c, \tilde{q});$$

which by integration implies the thesis. ■

Lemma 9 *Let $T > 0$. Under the assumptions (H1)–H(5) and for all $q \in \mathcal{C}_T$, the renewal equation (24) has a unique continuous solution $b(t; q), t \in [0, T]$. Moreover*

$$|b(t; q)| \leq \beta^* e^{\beta^* t}||p_0||_X, \; t \in [0, T],$$

and there exists an increasing function $K_b : [0, +\infty) \to [0, +\infty)$ such that for all $q, \tilde{q} \in \mathcal{C}_T$, $||q||_T, ||\tilde{q}||_T \leq \rho$, we have

$$|b(t; q) - b(t; \tilde{q})| \leq K_b(\rho)||p_0||_X ||q - \tilde{q}||_T.$$

Proof. The results follow by a suitable generalization of the standard approach presented in [17]. See the thesis (**??**) for details. ■

Theorem 10 *Let $T > 0$. Under the hypothesis (H1)–(H5), we have that the initial condition (9) uniquely fixes a nonnegative solution $p \in \mathcal{C}_T$ of (14), (15), and (16). Moreover*

$$||p(t, \cdot)||_X \leq e^{\beta * t}||p_0||_X, \; t \in [0, T],$$

$$||p(t, \cdot) - \tilde{p}(t, \cdot)||_X \leq e^{\ell(T)t}||p_0 - \tilde{p}_0||_X, \; t \in [0, T].$$

Proof. Define, for fixed $T > 0$ and $\rho > 0$, the set

$$\mathcal{K} = \{q \in \mathcal{C}_T | q(t, a) \geq 0 \; a.e. \; a \in [0, a_\dagger), t \in [0, T] \; ||q||_T \leq \rho\}$$

which is a closed set in $\mathcal{C}_T$, and the map $\mathcal{Q} : \mathcal{K} \subset \mathcal{C}_T \to \mathcal{C}_T$ given by

$$\mathcal{Q}q(t, a) = \begin{cases} \mathcal{U}(t, t - a; a - t, q(t, \cdot))b(t - a; q), & a.e. \; a \in (0, t) \\ \mathcal{U}(t, 0; a - t, q(t, \cdot))p_0(a - t), & a.e. \; a \in (t, a_\dagger) \end{cases}. \quad (26)$$

where $b(t;q)$ is the solution of the renewal equation (24) and p_0 satisfies (H7). We can prove that there exists $\rho > e^{\beta^* T}||p_0||_X$ such that $\mathcal{Q}$ maps $\mathcal{K}$ into itself and $\mathcal{Q}^N$ is a contraction on $\mathcal{K}$ for N sufficiently large. Thus there exists an unique positive solution $p \in \mathcal{C}_T$. ■

We conclude this section with some age-dependent models found in the literature, that can be represented in the general form(1)-(2)-(3).

Example 11 *In [17] the competition among juveniles and adults in a single population, i.e.* $d = 1$, *has been described by the Gurtin-MacCamy model*

$$\begin{cases} \frac{\partial p}{\partial t}(t,a) + \frac{\partial p}{\partial a}(t,a) + (\mu_0(a) + \mu_1(a, J(t), A(t)))p(t,a) = 0, & a \in [0, a_\dagger],\ t \geq 0, \\ p(t,0) = \int_0^{a_\dagger} \beta(a, J(t), A(t))p(t,a)da, & t \geq 0, \\ p(0,a) = p_0(a), & a \in [0, a_\dagger]. \end{cases} \tag{27}$$

where

$$S(t) = (J(t), A(t))^T$$

where

$$J(t) = \int_0^{a*} p(t,a)da,\ A(t) = \int_{a*}^{a_\dagger} p(t,a)da,\ t \geq 0$$

and $a*$ *is the maturation age.*

Example 12 *(see [17]). Let* $d = 2$. *In [17] the author considers an age-structured SI epidemic model described by*

$$\begin{cases} \frac{\partial p_1}{\partial t}(t,a) + \frac{\partial p_1}{\partial a}(t,a) + (\mu^{(0)}(a) + \mu^{(1)}(t,a)p_1(t,a) = 0, \\ \frac{\partial p_2}{\partial t}(t,a) + \frac{\partial p_2}{\partial a}(t,a) + (\mu^{(0)}(a) + \alpha)p_2(t,a) - \mu^{(1)}(t,a)p_1(t,a) = 0,\ a \in [0, a_\dagger],\ t \geq 0, \\ p_1(t,0) = \Phi(S(t))(\int_0^{a_\dagger} \beta(a)p_1(t,a)da, p_2(t,0) = 0, \quad t \geq 0, \\ p(0,a) = p_0(a), \quad a \in [0, a_\dagger]. \end{cases} \tag{28}$$

where

$$S(t) = \int_0^{a_\dagger} \gamma(a)(p_1(t,a) + p_2(t,a))da,\ t \geq 0. \tag{29}$$

Example 13 *In [1] the author propose the following models for a single population*

$$\begin{cases} \frac{\partial p}{\partial t}(t,a) + \frac{\partial p}{\partial a}(t,a) + (\mu^{(0)}(a) + \mu^{(1)}(t,a,S_1(t)))p(t,a) = 0,\ a \in [0, a_\dagger],\ t \geq 0, \\ p(t,0) = \int_0^{a_\dagger} \beta(t,a,S_2(t))p(t,a)da, \quad t \geq 0, \\ p(0,a) = p_0(a), \quad a \in [0, a_\dagger]. \end{cases} \tag{30}$$

2.1 Regular solutions

In the previous section we have seen that the solution exhibits a "smoothing effect" on the initial age distribution. Although the initial age distribution may be only integrable, the solution $p(t,a)$ regarded as a function of age a for a fixed time t is not only integrable on $[0, a_\dagger]$ but continuous on $[0, t]$. Since our aim is to consider numerical methods and in particular to test higher order numerical schemes to simulate the long-time behavior of the solutions, we now investigate regularity of the solutions. For all $T > 0$ and for some integer $k \geq 0$ we assume that

(A1) all the elements of the nonnegative matrix $\gamma(a)$ belong to $C^k([0, a_\dagger]; \mathbb{R})$ and have compact support;

(A2) all the elements of the nonnegative fertility matrix $\beta(t, a, S)$ belong to $\mathcal{C}^{k+1}([0, T] \times [0, a_\dagger] \times \mathbb{R}^n; \mathbb{R})$ with derivative with respect to S in $\mathcal{C}^k([0, T] \times [0, a_\dagger] \times \mathbb{R}^n; \mathbb{R})$ and bounded for S bounded. Moreover there exists an increasing function $\beta^* : [0, +\infty) \to [0, +\infty)$ such that

$$|\beta(t, a, S)| \leq \beta^*(|S|),\ a \in [0, a_\dagger], t \in [0, T], S \in \mathbb{R}^n;$$

(A3) the mortality matrix $\mu(t, a, S)$ is such that

- o (4) holds and thus it an be represented as the sum of the internal mortality diagonal matrix $\mu^{(0)}$ and the external mortality matrix $\mu^{(1)}$ as in (7);
- o the functions $\mu_i^{(0)} \in \mathcal{C}^{k+1}([0, a_\dagger); \mathbb{R}), i = 1, ..., d$, are nonnegative and moreover (10) holds;
- o all the elements of the external mortality matrix $\mu^{(1)}(t, a, S)$ belong to $C^{k+1}([0, T] \times [0, a_\dagger] \times \mathbb{R}^n; \mathbb{R})$ with derivative with respect to S in $C^k([0, T] \times [0, a_\dagger] \times \mathbb{R}^n; \mathbb{R})$ and bounded for S bounded;

(A4) for all $\rho > 0$, given $S \in \mathbb{R}^n, |S| \leq \rho$ and $a \in [0, a_\dagger), t \in [0, T]$

$$-\mu_i^{(1)}(t, a, S) + \textstyle\sum_{j=1, j\neq i}^{d} |\mu_{ij}(t, a, S)| \leq \mu_i^{(0)}(a),\ i = 1, ...d,$$

$$\mu_{ij}(t, a, S) \leq 0,\ i, j = 1, ..., d, i \neq j;$$

(A5) the initial function p_0 belongs to $C^k([0, a_\dagger]; \mathbb{R}^d)$, it is nonnegative and it has compact support.

Theorem 14 *Let $T > 0$. Under the assumptions (A1)–(A5) and the compatibility condition*

$$p_0(0) = \int_0^{a_\dagger} \beta(a, t, S_0) p_0(a) da,\ S_0 = \int_0^{a_\dagger} \gamma(a) p_0(a) da, \tag{31}$$

we have that the initial condition (9) uniquely fixes a nonnegative solution $p \in \mathcal{C}^0([0, T] \times [0, a_\dagger]; \mathbb{R}^d) \cap \mathcal{C}^{k+1}([0, T] \times [0, a_\dagger] - \{(t, a) | a = t\}; \mathbb{R}^d)$ of (1), (2), and (3).

Remark 15 *To ensure that the solution p belongs to $\mathcal{C}^k([0, T] \times [0, a_\dagger]; \mathbb{R}^d)$, for $k \geq 1$, by (15) we have to assume 31 and the further compatibility conditions on $p_0^{(\ell)}(0) = b^{(\ell)}(0)$ for $\ell = 1, ..., k$.*

Remark 16 *Observe that in (A5), we can assume*

$$\lim_{a \to a_\dagger} e^{\int_0^{a_\dagger} \mu^{(0)}(a) da} p_0(a) < +\infty$$

instead of the compact support condition.

3 Numerical methods

One of the features of the model (1) is that the mortality rate is separable in linear and nonlinear parts. Moreover, the linear part is diagonal and it is assumed to became unbounded as $a \to a_\dagger$ to take into account finite $a_\dagger$. This condition causes problems in the numerical integration, as pointed out by several authors (see [18], [21], [1]).

By applying the method of characteristic (MOC) or the method of lines (MOL) to the model (1), we get a system of differential-algebraic equations (DAEs) where the right-hand side of the differential equations consists of a linear part and a nonlin ear stiff part and the algebraic equations arise by the discretization of the new-born equations (3). Having long-time integration in mind, in section 3.1 we focus our attention on the MOL and on the efficient numerical integration of the resulting DAEs. The use of an implicit method allows to integrate efficiently the stiff part due to the unbounded mortality function.

Another possible line of investigation concerns the treatment of the linear and nonlinear part of the differential equations resulting by either MOL or MOC. In the literature different numerical approach have been analyzed: integrating factor (IF) methods, splitting methods, IMEX methods, Sliders, Exponential Time Dierencing methods (see [26] and the references therein). The idea of the IF method is to make a change of variable that allows us to solve for the linear part exactly, and then use a numerical scheme to solve the transformed, nonlinear equation. In section 3.1 we test the IF on the differential system arising by MOL (MOL-IF). The comparison between the various approach will be considered in the future.

Techniques that multiply the differential equations by an integrating factor and make a change of variable in order to efficiently solve the linear part and then choose a numerical scheme to integrate the transformed non linear equations has been used not only for ordinary differential equations (ODEs) but also for partial differential equations (PDEs) (see again [26] and the references therein). Thus in section 3.2 we propose the integrating factor (IF) approach for 1). We approximate again the modified model by MOL and we call the resulting approach IF-MOL. The IF approach requires the efficient computation of the survival probability, which is treated in section 3.2.1.

We assume here that the functions are sufficiently smooth to ensure that the solution is regular according the results given in theorem 14. To test the numerical methods we consider (1) in $[0, T]$ where

- **Example 17** *One population* $d = 1$ *(see [18])*
  - $\mu_0(a) = -\frac{\lambda}{a_\dagger - a}$, $\forall a \in [0, a_\dagger]$, $\lambda \in \mathbb{R}$
  - $\mu_1(t, a, S) = S + 1$, $\forall a \in [0, a_\dagger]$, $\forall t \in [0, T]$, $\forall \boldsymbol{S} \in \mathbb{R}$;
  - $\beta(t, a, S) = 5$, $\forall a \in [0, a_\dagger]$, $\forall t \in [0, T]$ $\forall \boldsymbol{S} \in \mathbb{R}$;
  - $\gamma(a) = 1$, $\forall a \in [a, a_\dagger]$;
  - $p_0(a) = e^{-5a} + \frac{e^{-5}}{4}, a \in [0, a^*], a^* < a_\dagger.$
- **Example 18** *Two species* $d = 2$
  - $\mu_0(a) = \begin{pmatrix} \frac{\lambda}{a_\dagger - a} & 0 \\ 0 & \frac{\lambda}{a_\dagger - a} \end{pmatrix}$
  - $\mu_1(a, t, S) = \begin{pmatrix} S_2 & 0 \\ -S_2 & 3 \end{pmatrix} \quad S = (S_1, S_2)^T$

- $\beta(a,t,S) = R_0\phi(S_1)\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}$ $S = (S_1,S_2)^T$, $R_0 = 6$ *and* $\phi(S_1) = 6 * max\{1 - \frac{S_1}{4}, 0\}$.
- $\gamma(a) = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$
- $p_0(t,a) = \begin{pmatrix}e^{-5a} + \frac{e^{-5}}{4},\ a \in [0,a^*], a^* < a_\dagger \\ a(a_\dagger - a)\ a \in [0,a_\dagger]\end{pmatrix}$

3.1 Direct approach: the method of lines

As first step we propose to integrate the resulting DAEs by DAE-solvers and we test them on our test problems 17 e 18. Let us restrict our attention to MOL and Consider $a_0 = 0 < a_1 < .... < a_M = a_\dagger$. By defining the functions

$$P_m(t) = P(t,a_m), m = 0,...,M,$$

we can construct an approximation as follows

$$p(t,a) \approx \sum_{m=0}^{M-1} \ell_m(a)P_m(t),\ a \in [0,a_\dagger],\ t \in [0,T], \tag{32}$$

where $\ell_m(t), m = 0,...,M$ are the Langrange polynomials relevant to the nodes a_m and $P_M(t) = 0$. By choosing the quadrature rule with nodes a_m and weights $w_m = \int_0^{a_\dagger} \ell_m(a)da$, we get the following DAEs

$$\begin{cases} P'_m(t) + \sum_{k=0}^{M-1} \ell'_k(a_m)P_k(t) + (\mu_0(a_m) + \mu_1(t,a_m,S_m(t))P_m(t) = 0, \\ m = 1,...,M-1 \\ \\ P_0(t) = \sum_{m=0}^{M-1} w_m\beta(t,a_m,S_m(t))P_m(t), \end{cases} \tag{33}$$

where $S_m(t) = \sum_{m=0}^{M-1} w_m\gamma(a_m)P_m(t)$ and the initial condition is given by

$$P_m(0) = p_0(a_m), m = 0,...,M-1. \tag{34}$$

Here we choose $a_m, m = 0,...,M$ as the Chebyshev extremal nodes and we propose to integrate in time (33)-(34) by numerical methods for index one DAEs (see [14]). The regularity of the solution and thus the order of accuracy depends on the compatibily condition (31). Therefore this approach is convenient when $T > a_\dagger$ and for the numerical simulation of long-time behaviour, as it has been confirmed by the numerical results.

Example 19 *We consider the problem (17) with* $a_\dagger = 1$ *and* $\lambda = 1$. *To test the performance of the MOL varing* M*, we integrate the resulting DAEs (33) in* $[0,T]$ *by MATLAB code ode15s with* $RelTol = AbsTol = TOL$ *where* $TOL = 10 * eps$*, where* $eps = 2^{-52}$. *The errors are estimated with respect to the values* $\tilde{p}(a_\dagger,0) = 11.94931346314318 \approx p(a_\dagger,0)$ *and* $\tilde{p}(2a_\dagger,0) = 13.25859019167530 \approx p(2a_\dagger,0)$*, which are obtained with* $M = 50$ *and* $TOL = 10*eps$. *This allows us to have an idea of the accuracy of the method, but in the future we will add a more complete analysis.*

M	steps in $[0, T]$	$p(T, 0)$	$absERR$	$relERR$
10	1602	11.94960428263432	$2.9082e-04$	$2.4338e-05$
20	3047	11.94938478880748	$7.1326e-05$	$5.9690e-06$
40	5793	11.94932148078744	$8.0176e-06$	$6.7097e-07$

Table 1: Numerical results for (17) on $[0, T]$, with $T = a_{\dagger} = 1s$ and $TOL = 10 * eps$.

M	steps in $[0, T]$	$p(T, 0)$	$absERR$	$relERR$
10	2001	13.25861519778685	$2.5006e-05$	$1.8860e-06$
20	3706	13.25859610672555	$5.9151e-06$	$4.4613e-07$
40	6925	13.25859086340420	$6.7173e-07$	$5.0664e-08$

Table 2: Numerical results for (17) on $[0, T]$, with $T = 2a_{\dagger}$ and $TOL = 10 * eps$

By comparing the number of steps in tables 1-2, it is clear that the bigger computational effort is devoted to the integration in the interval $[0, T]$, where the solution is less accurate. In table 3 we integrate the equations for $t \in [T, 2T]$, $T = a_{\dagger}$, by choosing as initial solution $p_0(T, a)$ the polynomial interpolating the computed values with $M = 50$. The number of steps has been reduced and the accuracy has been improved, confirming that the MOL with Chebyshev nodes could be use for long time simulations.

M	steps in $[T, 2T]$	$p(0, 2a_{\dagger})$	$absERR$	$relERR$
10	479	13.25859130588390	$1.1142e-06$	$8.4037e-08$
20	706	13.25859025243245	$6.0757e-08$	$4.5825e-09$
40	1180	13.25859019099947	$6.7576e-10$	$5.0968e-11$

Table 3: Numerical results for (17) on $[T, 2T]$ with $T = a_{\dagger}$ and $TOL = 10 * eps$

By relaxing the tolerance TOL, we obtain the results in tables 4- 5, which confirm the good performance of the DAE solver.

Example 20 *We consider the model 18 with $a_{\dagger} = 1$ and $\lambda = 1$. The resulting DAEs (33) is solved in $[0, T]$ by the Matlab code ode15s with $TOL = 10*eps$ and we estimate the errors with respect to the values $\tilde{p_1}(0, a_{\dagger}) = 5.262131770424491 \approx p_1(0, a_{\dagger})$ and $\tilde{p_1}(2a_{\dagger}, 0) = 5.328917335996803 \approx p_1(2a_{\dagger}, 0)$ respectively, which are obtained with $M = 50$ and $TOL = 10 * eps$. We have that $\tilde{p_2}(0, a_{\dagger}) = 0$ and $\tilde{p_1}(2a_{\dagger}, 0) = 0 \approx p_2(2a_{\dagger}, 0)$.*

The comparison of the number of steps in tables 6-7 confirms again that the approach is convenient for the long-time simulations. In tables 8-9 we study as the cost reduces with $TOL = 10^{-6}$.

3.1.1 MOL-IF aproach

The right-hand side of the differential equations in (33) consists of a non linear part due to the mortality matrix μ_1 and of a linear part due to either the differentiation matrix $D_M := (\ell_k'(a_m))_{mk}, m, k = 0, ..., M-1$ and the intrinsic mortality matrix

M	steps	$p(T,0)$	$absERR$	$relERR$
20	150	11.94938473364693	$7.1270e-05$	$5.9644e-06$
40	287	11.94932144698652	$7.9838e-06$	$6.6814e-07$

Table 4: Numerical results for (17) on $[0,T]$, with $T = a_\dagger$ and $TOL = 10^{-6}$

N	steps in $[0,T]$	valore $p(T,0)$	$ERRabs$	$ERRrel$
20	178	13.25857386774502	$1.6324e-05$	$1.2312e-06$
40	341	13.25858809892424	$2.0927e-06$	$1.5784e-07$

Table 5: Numerical results for (17) on $[0,T]$, with $T = 2a_\dagger$ and $TOL = 10^{-6}$

$M_0 = diag(\mu_0(a_m))$. By defining $\mathbf{P}(t) := (P_1(t), ..., P_{M-1}(t))^T$, the differential equations in (33) can be rewritten as

$$\mathbf{P}'(t) = -M_0\mathbf{P}(t) + N(t, \mathbf{P}(t)), \tag{35}$$

where

$$N(t, \mathbf{P}) = -diag(\mu_1(t, a_m, \textstyle\sum_{m=0}^{M-1} w_m \gamma(a_m) P_m)\mathbf{P} - D_M\mathbf{P}, \tag{36}$$

Multiplying both sides of (35) by the integrating factor

$$e^{M_0 t}$$

and, by defining

$$\mathbf{U}(t) = e^{M_0 t}\mathbf{P}(t),$$

we get

$$\mathbf{U}'(t) = e^{M_0 t} N(t, e^{-M_0 t}\mathbf{U}(t)) \tag{37}$$

Observe by this choice the integrating factor is a diagonal matrix, so that scalars rather than matrices are involved.

3.2 Reformulation of the model and the IF approach

The idea is to make a change of variable that allows us to solve the linear part exactly or with a proper algorithm, and then concentrate the effort in the approximation to solve the transformed nonlin ear equation. This technique has been used for PDEs (see [26] and the references therein). When it is applied to ODEs is called Integrating Factor (IF) approach and we use the same acronym for the method applied to our model. Let us define

$$u = \pi^{-1} p$$

where $\pi(t, a)$, called the **integrating factor**, is the solution of the matrix equation

$$\begin{aligned}
&\tfrac{\partial \pi}{\partial t}(t,a) + \tfrac{\partial \pi}{\partial a}(t,a) + \mu_0(a)\pi(t,a) = 0,\ a \in [0, a_\dagger],\ t \geq 0, \\
&\pi(t,0) = I_d,\ t \geq 0, \\
&\pi(0,a) = I_d,\ a \in [0, a_\dagger),
\end{aligned} \tag{38}$$

M	steps in $[0,T]$	$p_1(T,0)$	$absERR$	$relERR$
10	2230	5.261790545481460	$3.4122e-04$	$6.4845e-05$
20	4084	5.262092072498592	$3.9698e-05$	$7.5441e-06$
40	7414	5.262132031697793	$2.6127e-07$	$4.9652e-08$

Table 6: Numerical results for (18) on $[0,T]$, with $T=a_\dagger$ and $TOL=10*eps$.

M	steps in $[0,T]$	$p_1(T,0)$	$absERR$	$relERR$
10	3081	5.328915278902983	$2.0571e-06$	$3.8602e-07$
20	5455	5.328917326261293	$9.7355e-09$	$1.8269e-09$
40	9672	5.328917356690360	$2.0694e-08$	$3.8833e-09$

Table 7: Numerical results for (18) on $[0,T]$ with $T=2a_\dagger$ and $TOL=10*eps$.

with I_d the identity matrix. By defining the exponential matrix

$$\pi(a)=e^{-\int_0^a \mu_0(s)ds},\ a\in[0,a_\dagger], \tag{39}$$

we get

$$\pi(t,a)=\begin{cases}\pi^{-1}(a-t)\pi(a),\ a\geq\ t,\\ \pi(a),\ a\leq\ t.\end{cases} \tag{40}$$

Remark 21 *Since μ_0 is assumed to be a diagonal matrix, the problem (38) reduces to the solution of the d-scalar equations*

$$\begin{cases}\pi_i'(a)=-\mu_{0,i}(a)\pi_i(a),\ a\in[0,a_\dagger],\quad i=1,...,d\\ \pi_i(0)=1,\end{cases} \tag{41}$$

that, in some cases, can be done exactly. Moreover observe that our assumptions on the internal mortality matrix, ensure that $\pi_i(a_\dagger)=e^{-\int_0^{a_\dagger}\mu_i(s)ds}=0, i=1,...,d$ and $\pi_i(a_\dagger,t)=0,\ t\geq a_\dagger$.

The equations to solve to find u are given by

$$\begin{aligned}&\frac{\partial u}{\partial t}(t,a)+\frac{\partial u}{\partial a}(t,a)+\bar{\mu}(t,a,\bar{S}(t))u(t,a)=0,\ a\in[0,a_\dagger),\ t\geq0,\\ &u(t,0)=\int_0^{a_\dagger}\bar{\beta}(a,t,\bar{S}(t))u(t,a)da,\ t\geq0,\\ &u(0,a)=p_0(a),\ a\in[0,a_\dagger],\\ &\bar{S}(t)=\int_0^{a_\dagger}\bar{\gamma}(t,a)u(t,a)da,\ t\geq0.\end{aligned} \tag{42}$$

where

$$\begin{aligned}&\bar{\mu}(a,t,S)=\pi^{-1}(t,a)\mu_1(a,t,S)\pi(t,a)\\ &\bar{\beta}(a,t,S)=\beta(a,t,S)\pi(t,a)\\ &\bar{\gamma}(t,a)=\gamma(a)\pi(t,a).\end{aligned} \tag{43}$$

This reformulation allows also to obtain existence, uniqueness and regularity results. It is important to observe that $\bar{\mu}_{ii}(a,t,S)=(\mu_1)_{ii}(a,t,S), i=1,...d$ while $(\mu_1)_{ij}(a,t,S)$ is the solution of $\pi_i(t,a)\bar{\mu}_{ij}(a,t,S)=(\mu_1)_{ij}(a,t,S)\pi_j(t,a), i,j=1,...d, i\neq j$. In general such equations are not well defined when $t\geq a_\dagger, a=a_\dagger$.

M	steps in $[0,T]$	$p_1(T,0)$	$absERR$	$relERR$
20	191	5.262091744435034	$4.0026e-05$	$7.6064e-06$
40	335	5.262131865607987	$9.5183e-08$	$1.8088e-08$

Table 8: Numerical results for (18) on $[0,T]$ with $T = a_\dagger$ and $TOL = 10^{-6}$

M	steps in $[0,T]$	$p_1(T,0)$	$absERR$	$relERR$
20	259	5.328917235325827	$1.0067e-07$	$1.8891e-08$
40	458	5.328917624916919	$2.8892e-07$	$5.4217e-08$

Table 9: Numerical results for (18) on $[0,T]$ with $T = 2a_\dagger$ and $TOL = 10^{-6}$

But in the scalar case, i.e. $d = 1$, or when the matrices μ_1 and π commutes, we have

$$\bar{\mu}(a,t,S)) = \mu(a,t,S). \tag{44}$$

In epidemic model (see the S-I model in (28)) where p_i represents a class of the same population and thus the intrinsic mortality rates are equal, we obtain that $\bar{\mu}(a,t,S)) = \mu(a,t,S)$. For instance for the S-I model (28) we have that

$$\bar{\mu}(t,a) = \begin{pmatrix} -\lambda(t,a) & 0 \\ -\lambda(t,a)e^{\alpha a} & 0 \end{pmatrix}. \tag{45}$$

In the general case, since the survival probability of the single species goes to zero, we can solve the problem in the interval $[0,\bar{a}]$ with $\bar{a} < a_\dagger$ defined by survival probability greater than a fixed threshold. Further comments can be found in section **??**. In that follows, we assume that the matrix $\bar{\mu}$ is well defined and bounded.

To solve numerically (42) we need an ODE solver for the survival probability π in (39), a PDE solver for hyperbolic equations together a quadrature rule to approximate the size and the boundary conditions. The literature is wide (see ...). Here we focus the attention on RK-methods to approximate the survival probability and on the method of line (MOL) based on Chebyshev nodes to solve (42) for long time simulations.

Dire che lavorerereremo con eta' massima minore o uaguale di a che diversi approcci saranno nalizzatiscelta.

3.2.1 Approximation of the survival probability

As observed in remark 21, the integrating factor can be obtained solving d scalar ordinary differential equations 41 and efficient numerical methods are necessary. Here we propose the s-stage Runge-Kutta methods of Radau IIA type, which have order $p = 2s - 1$, are AN-stable and provide a continuos approximation of the solutions of uniform order $q = s + 1$ (see [14]). In the following tables, we present the numerical results obtained on the test problem

$$y'(a) = -\frac{\lambda}{1-a}y(a), \ a \in [0,\bar{a}], \tag{46}$$

where $\bar{a} < a_\dagger = 1$ and $\lambda \in \mathbb{R}$ is a parameter. In such case the solution of (46) can be obtained exactly

$$y(a) = (1-a)^\lambda. \tag{47}$$

The equation (46) was introduced in [18], where the authors show that the effective rate of convergence is the theoretical one for Eulers method and for Crank-Nicolson, whenever λ is at least as large as the theoretical asymptotic order of convergence. When the value of λ is smaller than that rate, then the effective rate of convergence is actually given by λ. This means that the method degenerates" and it does not converge any longer at its theoretical rate. Here we prove that by choosing in a clever way $a_f < 1$, we could preserve the convergence order with constant stepsize for $s = 2, 3$ and by using RADAU5 with step size control [14] we can obtain the machine precision accuracy for different value of λ, i.e. $\lambda = 0.5, 1.5$

h	s	err	s	err
0.1	2	6.4186e-04	3	1.6929 e-05
0.05		9.5154e-05		7.8726 e-07
0.025		1.2585e-05		2.8454 e-08
0.01		8.1983e-07		3.0582 e-10
0.005		1.0275e-07		9.6267e-12
0.0025		1.2582 e-08		3.072 e-13
0.001		8.2266 e-10		2.2204 e-15

Table 10: $a_f = 0.9$ and $\lambda = 0.5$.

3.2.2 IF-MOL method

Consider $a_0 = 0 < a_1 < .... < a_M = a_\dagger$ and define the functions $U_m(t) = u(a_m, t), m = 0, ..., M$. We consider

$$u(t, a) \approx \sum_{m=0}^{M} \ell_m(a) U_m(t),\ a \in [0, a_\dagger],\ t \in [0, T], \tag{48}$$

where $\ell_m, m = 0, ..., M$ are the Langrange polynomials relevant to the nodes a_m. By choosing the quadrature rule with nodes a_m and weights $w_m = \int_0^{a_\dagger} \ell_m(a)da$, we get from (42) the following differential algebraic equations (DAEs)

$$\begin{array}{l} U_m' + \sum_{k=0}^{M} \ell_k'(a_m) U_k(t) + \bar{m}(t, a_m, \bar{S}_m(t)) U_m = 0,\ m = 1, ..., M \\ U_0(t) = \sum_{m=0}^{M} w_m \bar{\beta}(t, a_m, \bar{S}_m(t)) U_m(t), \end{array} \tag{49}$$

and

$$\begin{array}{l} U_m(0) = p_0(a_m), m = 0, ...M, \\ U_M(t) = 0 \\ \\ \bar{S}_m(t) = \sum_{m=0}^{M} w_m \bar{\gamma}(a_m) U_m(t). \end{array} \tag{50}$$

The regularity of the solution and thus the order of accuracy a depends on the splicing conditions. Therefore this approach is more convenient for $t > a_\dagger$ and for long time behaviour. Concerning the integration in time of (49)-(50) we use numerical methods suitable forindex one DAEs (see Hairer, MatlabCode). In the following example we present the numerical results for (17). To sake of brevity, we omit the results obtained on (18) since they are analogous of the following ones.

Example 22 *We consider the problem (17) with $a_\dagger = 1$ and $\lambda = 1$. To test the performance of the IF-MOL varing M, we integrate the resulting DAEs (49) in $[0, T]$ by MATLAB code ode15s with $RelTol = AbsTol = TOL$ where $TOL = 10{*}eps$. Moreover we consider the exact integrating factor represantation.The errors are estimated*

M	steps in $[0, T]$	$p(T, 0)$	$absERR$	$relERR$
10	3634	11.94942319390904	$1.0973e-04$	$9.1830e-06$
20	6822	11.94931782260240	$4.3596e-06$	$3.6483e-07$
40	12735	11.94930116680950	$1.2296e-05$	$1.0290e-06$

Table 11: Numerical results for (17) on $[0, T]$, with $T = a_\dagger = 1s$ and $TOL = 10 * eps$.

M	steps in $[0, T]$	$p(T, 0)$	$absERR$	$relERR$
10	4338	13.25859811502223	$7.9233e-06$	$5.9760e-07$
20	7946	13.25859015271002	$3.8965e-08$	$2.9389e-09$
40	14612	13.258590148882	$4.2793e-08$	$3.2276e-09$

Table 12: Numerical results for (17) on $[0, T]$, with $T = 2a_\dagger$ and $TOL = 10 * eps$

with respect to the values in ().

The results in tables 14, 15 and 13 suggest the same comments in Example 22. In particular the number of steps confirm that the bigger computational effort is devoted to the integration in the first interval where the solution is less accurate and that the approach is good for long time simulations. But with respect to the direct approach, the IF-MOL one requires more steps and so costs more, also for $TOL = 10^{-6}$ in tables 14- 15. Concerning the accuracy, the results seem better in the direct one, but a more accurate analysis is necessary to support any conclusions. Nevertheless we recall that in general the error in IF-MOL approach depends also to the error in the integrating factor approximation.

M	steps in $[T, 2T]$	$p(0, 2a_\dagger)$	$absERR$	$relERR$
10	766	13.25859185943840	$1.6678e-06$	$1.2579e-07$
20	1088	13.25858937982010	$8.1186e-07$	$6.1232e-08$
40	1998	13.2585890274612	$1.1642e-06$	$8.7808e-08$

Table 13: Numerical results for (17) on $[T, 2T]$ with $T = a_\dagger$ and $TOL = 10 * eps$

4 Conclusion

In the paper we proposed a nonlinear model to describe the dynamics of more populations or interacting species of the same population, which generalizes the Gurtin-MacCamy one. We gave results on the existence, the uniqueness and the regularity of the solutions. In the second part we considered a numerical approach, which is based on the discretization of the the age part by the method of lines on the Chebyshev points and on the integration of the resulting DAEs by a suitable solver. The idea is to have a high-order approximations of the solutions, which is a desirable feature for long time simulations. The numerical results on the test problems gave a first insight on the performances and allowed to conclude that MOL direct approach is better than the MOL-IF one. We also proposed an implicit numerical method to integrate efficiently the survival probability.

M	steps	$p(T,0)$	$absERR$	$relERR$
20	391	11.94931815219286	$4.6890e-06$	$3.92416e-07$
40	622	11.94930128784704	$1.217e-05$	$1.0189e-06$

Table 14: Numerical results for (17) on $[0,T]$, with $T=a_\dagger$ and $TOL=10^{-6}$

N	steps in $[0,T]$	valore $p(T,0)$	$ERRabs$	$ERRrel$
20	450	13.25858826247615	$1.9292e-06$	$1.4550e-07$
40	730	13.25858846734374	$1.7243e-06$	$1.3005e-07$

Table 15: Numerical results for (17) on $[0,T]$, with $T=2a_\dagger$ and $TOL=10^{-6}$

Moreover we obtained some suggestions for future lines of investigations. We will consider

- the improvement of the accuracy when $0 \leq t \leq a_\dagger$;
- the development of a specific DAE solver which takes into account the linear and nonlinear parts;
- the error analysis of the method;
- the evaluation the the method's efficiency and accuracy by means of more test problems.

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